3.2 Inverse Functions and Logarithms

Two functions which "undo" each other are said to be inverse functions.

Just as some actions cannot be undone, (e.g. once you cook an egg, you can't uncook it), not every function has an inverse function. In order for a function to have an inverse, the function must be one-to-one, which means that every output (y-value) must be paired with exactly one input (x-value).

**Definition** A function $f$ is called a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

The easiest way to check this is by using the horizontal line test.

**Horizontal Line Test** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Determine whether or not each function below is one-to-one.
**DEFINITION** Let $f$ be a one-to-one function with domain $A$ and range $B$. Then its inverse function $f^{-1}$ has domain $B$ and range $A$ and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any $y$ in $B$.

**Caution:** Do not confuse the superscript -1 in the notation for $f$ inverse with an exponent. $f^{-1}(x)$ is not the same as $[f(x)]^{-1}$.

Example: Given $f(1) = 7$, $f(3) = 4$, and $f(5) = -6$, find each of the following:

- $f^{-1}(4)$
- $f^{-1}(7)$
- $f^{-1}(-6)$

The diagram below makes the relationship between $f$ and $f^{-1}$ clear.

Example: Given $f(x) = 3x - 4$, find $f^{-1}(6)$ without finding $f^{-1}(x)$. 
The following two cancellation equations hold for $f$ and $f^{-1}$.

$$f^{-1}(f(x)) = x \text{ for every } x \text{ in } A.$$

$$f(f^{-1}(x)) = x \text{ for every } x \text{ in } B.$$

These are basically just restatements of my original claim that inverse functions are functions which "undo" each other. Whatever $f$ does to $x$, $f^{-1}$ undoes, and vice-versa.

To find an inverse function, write $f(x)$ as $y$, switch $x$ and $y$, and solve for $y$, then write $y$ as $f^{-1}(x)$.

Ex: Given $f(x) = \sqrt[3]{7x - 4}$ find $f^{-1}(x)$. 

Note that since the point \((a, b)\) is on the graph of \(f(x)\) if and only if the point \((b, a)\) is on the graph of \(f^{-1}(x)\), the graph of \(f^{-1}\) is a reflection of the graph of \(f\) across the line \(y = x\).

Example: Sketch the graphs of \(f(x) = 3x - 6\) and its inverse below.

![Graph of \(f(x) = 3x - 6\) and its inverse](image)

Note that both \(f(x)\) and \(f^{-1}(x)\) are continuous functions. This suggests the following theorem (which is proved in Appendix D).

**THEOREM** If \(f\) is a one-to-one continuous function defined on an interval, then its inverse function \(f^{-1}\) is also continuous.

What about differentiability? If \(f(x)\) is differentiable everywhere, then it will not have any corners or breaks in it. Thus it is reasonable to expect that its inverse will not have any corners or breaks. However, note that the slopes of the tangent lines to \(f\) and \(f^{-1}\) are reciprocals.

\[
(f^{-1})'(a) = \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} = \frac{1}{f'(b)}
\]

Note that if \(f(b) = a\), then \(b = f^{-1}(a)\), which leads directly to theorem 7.

**THEOREM** If \(f\) is a one-to-one differentiable function with inverse function \(f^{-1}\) and \(f'(f^{-1}(a)) \neq 0\), then the inverse function is differentiable at \(a\) and

\[
(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}
\]

However, if the tangent to \(f(x)\) has slope 0 at the point \((b, a)\), (i.e., \(f'(b) = f'(f^{-1}(a)) = 0\)), the tangent to \(f^{-1}(x)\) would be undefined at \((a, b)\). Thus \(f^{-1}(x)\) would not be differentiable at that point.
Use theorem 7 to do the following:

a) Given $f(x) = x^3$, find $(f^{-1})'(8)$.

First note that $f^{-1}(8) = b$, such that $f(b) = 8$, so our first task is to find $b$.

b) Given $f(x) = 2x + 7\cos x + 4\sin x$, find $(f^{-1})'(7)$.

Again, our first task is to find the number $b$ for which $f(b) = 7$. 
Logarithmic Functions:

Recall that the exponential function \( f(x) = a^x \) is either an increasing or a decreasing function, and so by the horizontal line test, it is a one-to-one function, and thus has an inverse. Its inverse is the logarithmic function with base \( a \), denoted \( f^{-1}(x) = \log_a x \).

Note that \( \log_a x = y \) iff \( a^y = x \). This leads to the following cancellation equations:

\[
\log_a a^x = x \quad \text{for all } x \in \mathbb{R} \quad a^{\log_a x} = x \quad \text{for all } x > 0
\]

Since the logarithm function is the inverse of the exponential function, its domain is the same as the range of the exponential's, and its range is the same as the domain of the exponential's. Thus the function \( y = \log_a x \) has domain \((0, \infty)\) and range \((-\infty, \infty)\).

The following laws of logarithms are related to the laws of exponents, and can be used to expand or condense logarithm expressions.

**LAWS OF LOGARITHMS**  If \( x \) and \( y \) are positive numbers, then

1. \( \log_a(xy) = \log_a x + \log_a y \)

2. \( \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \)

3. \( \log_a(x^r) = r \log_a x \quad \text{(where } r \text{ is any real number)} \)

Example: Use the laws of logarithms to evaluate \( \log_5 4 + \log_5 50 - \log_5 8 \)
Note that the graph of the logarithmic function with base $a$ would be the reflection of the graph of the exponential function with base $a$ across the line $y = x$. The limits of exponential functions from 3.1 are related to the following limits of logarithmic functions.

$$\lim_{{x \to \infty}} \log_a x = \infty \quad \text{and} \quad \lim_{{x \to 0^+}} \log_a x = -\infty$$

Note that while the graph of the exponential function has a horizontal asymptote at $y = 0$, the graph of the logarithmic function has a vertical asymptote at $x = 0$.

Of all possible bases for the logarithmic function, it turns out that the most useful one is base $e$, which was defined in the previous section. The logarithm with base $e$ is referred to as the **natural logarithm** and is denoted $\ln x$.

$$\log_e x = \ln x$$

Replacing the $a$'s in the earlier equivalence statement leads us to the following:

$$\ln x = y \quad \text{iff} \quad e^y = x$$

| $\ln(e^x) = x$ | $x \in \mathbb{R}$ |
| $e^{\ln x} = x$ | $x > 0$ |

Note that if $x = 1$, the first statement becomes $\ln e = 1$. 

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These properties allow us to solve both logarithmic equations and exponential equations, as in the following examples:

\[
\ln(e^x) = x \quad x \in \mathbb{R} \\
e^{\ln x} = x \quad x > 0
\]

They also assist us in finding inverse functions for exponential and logarithmic functions, as in the following examples. Find \( f^{-1}(x) \) for each of the following.

\[
f(x) = 2e^{x+5} \quad f(x) = \ln(4x - 3)
\]
A logarithm with any base can be expressed as a natural logarithm by using the change of base formula.

**Change of Base Formula** For any positive number $a$ ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Express each of the following logarithms as a ratio of natural logarithms.

$$\log_7 20 \quad \log_{0.5} 1$$