3.1 Exponential Functions

An exponential function \( f \) with base \( a \) is defined by \( f(x) = a^x \) where \( a \) is a positive constant and \( x \) is any real number.

If \( x = n \), a positive integer, then

\[
a^n = a \cdot a \cdot \cdots \cdot a
\]

\( n \) factors

If \( x = 0 \), then \( a^0 = 1 \), and if \( x = -n \), where \( n \) is a positive integer, then

\[
a^{-n} = \frac{1}{a^n}
\]

If \( x \) is a rational number, \( x = p/q \), where \( p \) and \( q \) are integers and \( q > 0 \), then

\[
a^x = a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p
\]

It can be shown that \( x \) can be any real number. Since every irrational number can be located between two rational numbers, (e.g., \( 1.7 < \sqrt{3} < 1.8 \)), it stands to reason that \( a^x \) would fall between two rational numbers (e.g. \( a^{1.7} < \sqrt{3} < a^{1.8} \)) and so would have a place on the graph for any value of \( x \).

Figure 6 shows that there are basically three cases of exponential functions. (However, note that case 2 is not actually an exponential function.)

![Graphs of exponential functions](image)
The properties of the exponential function are summarized in the following theorem.

1. \( a^{x+y} = a^x a^y \)  
2. \( a^{x-y} = \frac{a^x}{a^y} \)  
3. \( (a^x)^y = a^{xy} \)  
4. \( (ab)^x = a^x b^x \)

Note that unless the graph of \( y = a^x \) has a transformation applied to it, it will have \( y \)-intercept \((0, 1)\) and will not intersect the \( x \)-axis. In fact, it will have a horizontal asymptote at \( y = 0 \). Also note that for \( a > 1 \), larger values of \( a \) give more rapidly increasing graphs. For \( 0 < a < 1 \), smaller values of \( a \) give more rapidly decreasing graphs. Furthermore,

If \( a > 1 \), then \( \lim_{x \to -\infty} a^x = 0 \) and \( \lim_{x \to \infty} a^x = \infty \)

If \( 0 < a < 1 \), then \( \lim_{x \to -\infty} a^x = \infty \) and \( \lim_{x \to \infty} a^x = 0 \)

Just as with the graphs of other types of functions, the graphs of exponential functions can have transformations applied to them, including shifts, reflections, stretches, and compressions.

Describe how the graph of each function below would differ from the graph of \( y = a^x \). Also state the horizontal asymptote for each graph.

\[ y = a^{x-3} \quad y = -2a^x \quad y = a^x+5 \]
Determine a formula for each exponential function $y = C \cdot a^x$ whose graph is shown below.
If $a > 1$, then  \[ \lim_{x \to \infty} a^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} a^x = 0 \]

If $0 < a < 1$, then  \[ \lim_{x \to \infty} a^x = 0 \quad \text{and} \quad \lim_{x \to -\infty} a^x = \infty \]

Use the properties above (and your knowledge of the limit laws) to find each of the following limits.

\[ \lim_{x \to -\infty} (5^x - 3) \quad \quad \lim_{x \to \infty} (-7^x) \]

\[ \lim_{x \to -\infty} (5(1.2)^{x+3}) \quad \quad \lim_{x \to \infty} \left( \frac{1}{3} \right)^x + 2 \]
The most convenient base of all for an exponential function (for many reasons) is the natural base $e$, which is defined as follows:

$$e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$$

The graph of the function $y = (1 + x)^{1/x}$ is shown in figure 8. Note that it is not defined at $x = 0$, but its behavior near 0 is shown in the table below. These values suggest that the limit above exists and that $e \approx 2.71828181$

<table>
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<th>$x$</th>
<th>$(1 + x)^{1/x}$</th>
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<tr>
<td>0.1</td>
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</tr>
<tr>
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<td>2.70481383</td>
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<tr>
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<td>0.00000001</td>
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</table>
The function \( y = e^x \) is called the natural exponential function. Because \( e \) lies between 2 and 3, the graph of \( y = e^x \) lies between the graphs of \( y = 2^x \) and \( y = 3^x \).

The exponential function \( f(x) = e^x \) is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph and properties, which are summarized below.

**PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION**  The exponential function \( f(x) = e^x \) is a continuous function with domain \( \mathbb{R} \) and range \((0, \infty)\). Thus \( e^x > 0 \) for all \( x \). Also

\[
\lim_{x \to -\infty} e^x = 0 \quad \lim_{x \to \infty} e^x = \infty
\]

So the \( x \)-axis is a horizontal asymptote of \( f(x) = e^x \).

Use the properties above (and your understanding of limits) to find each of the following limits.

\[
\lim_{x \to \infty} \frac{e^x}{e^x - 1} \quad \lim_{x \to \infty} \frac{e^{5x} - 2}{e^x}
\]
Write the equation for the graph that will result from the following transformations to the graph of $y = e^x$.

(a) shifting it 3 units to the left and 4 units down

(b) vertically stretching it by a factor of 5 and reflecting it across the $x$-axis.

State the domain of each of the following functions.

$$
y = \frac{e^x}{e^x - 1} \quad \quad \quad \quad \quad \quad \quad \quad \quad y = \frac{1 - x}{e^{2x+5}}$$