1.6 Limits Involving Infinity

In this section we investigate the global behavior of functions; in particular, whether their graphs approach vertical or horizontal asymptotes. We will also introduce a new way of expressing the fact that a function goes to infinity as a value of $x$ approaches a particular number, as in the following example.

\[
\lim_{x \to 0} \frac{1}{x^2}
\]

Clearly, the function does not approach a number as $x$ approaches 0, but the function approaches infinity from the left and the right of 0, so we use notation

\[
\lim_{x \to 0} \frac{1}{x^2} = \infty
\]

**DEFINITION** The notation

\[
\lim_{x \to a} f(x) = \infty
\]

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking $x$ sufficiently close to $a$ (on either side of $a$) but not equal to $a$.

Another notation for this is: \( f(x) \to \infty \) as \( x \to a \).
Consider the following examples:

\[ f(x) = \frac{x^2 - 9}{x^2 - 4} \]

\[ f(x) = \frac{2x - 8}{x^2 - 5x + 6} \]
Find the vertical asymptotes of \( y = \tan x \).

In all of the limits we have considered so far, we have looked at what is happening to \( f(x) \) as \( x \) approaches a number. Now we consider what happens to \( f(x) \) as \( x \) gets larger and larger, i.e. as \( x \) approaches infinity (or negative infinity).

We will begin with a numerical approach (as in 1.3) for the following function:

\[
f(x) = \frac{x^2 - 9}{x^2 - 25}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>+1</th>
<th>+10</th>
<th>+100</th>
<th>+1000</th>
<th>+10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Another notation for this is: \( f(x) \to L \) as \( x \to \infty \).
Find the infinite limits, limits at infinity, and asymptotes for the function $f$ whose graph is shown below.

vertical asymptotes (va):

horizontal asymptotes (ha):

Most of the limit laws from section 1.4 also hold for limits at infinity. It can be proved that all of them except Laws 9 and 10 are also valid if $x \to a$ is replaced by $x \to \infty$ or $x \to -\infty$.

In particular, if we combine Law 6 with the results of Example 4, which shows that $\lim_{x \to \infty} \frac{1}{x} = 0$ and $\lim_{x \to -\infty} \frac{1}{x} = 0$, we get the following:

If $n$ is a positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x^n} = 0$$
Sketch the graph of an example of a function $f$ that satisfies all of the given conditions.

\[
\lim_{x \to 2} f(x) = \infty, \quad \lim_{x \to -2^{-}} f(x) = \infty, \quad \lim_{x \to -2^{+}} f(x) = -\infty, \quad \lim_{x \to -\infty} f(x) = 0, \quad \lim_{x \to \infty} f(x) = 0, \quad f(0) = 0
\]

\[
\lim_{x \to -3} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = 2, \quad f(0) = 0, \quad f \text{ is even}
\]
Evaluate each of the following, if they exist.

\[
\lim_{x \to 2} \frac{x + 5}{x - 2} \quad \lim_{x \to \infty} \frac{x + 5}{x - 2} \quad \lim_{x \to \infty} 5\sin(2x)
\]

\[
\lim_{x \to \infty} \frac{4}{(x - 7)^2} \quad \lim_{x \to \infty} x^3 \quad \lim_{x \to \infty} \frac{4x^3 + x}{x^2 + 1}
\]
Find a formula for the function $f$ that satisfies the following conditions:

\[
\lim_{x \to \pm \infty} f(x) = 0, \quad \lim_{x \to -1^-} f(x) = -\infty,
\]
\[
\lim_{x \to -1^+} f(x) = \infty, \quad \lim_{x \to 6^-} f(x) = -\infty,
\]
\[
\lim_{x \to 6^+} f(x) = \infty, \quad f(4) = 0
\]