A Reform Approach to Business Calculus

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PREFACE

This supplement consists of my lectures of a freshmen-level mathematics class offered at Arkansas Tech University. The lectures are designed to accompany the textbook "*Applied Calculus*" written by Hughes-Hallett et al.
The lectures cover Chapters 1, 2, 3, 4, 5, 6, and 7. These chapters are basically well suited for a one semester course in Business Calculus.

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1 Functions and Function Notation

Functions play a crucial role in mathematics. A function describes how one quantity depends on others. More precisely, when we say that a quantity y is a **function** of a quantity x we mean a rule that assigns to every possible value of x exactly one value of y. We call x the **input** and y the **output**. In **function notation** we write

$$y = f(x)$$

Since y depends on x it makes sense to call x the **independent variable** and y the **dependent variable**.
In applications of mathematics, functions are often representations of real world phenomena. Thus, the functions in this case are referred to as **mathematical models.** If the set of input values is a finite set then the models are known as **discrete** models. Otherwise, the models are known as **continuous** models. For example, if H represents the temperature after t hours for a specific day, then H is a discrete model. If A is the area of a circle of radius r then A is a continuous model.
There are four common ways in which functions are presented and used: By words, by tables, by graphs, and by formulas.

Example 1.1
The sales tax on an item is 6%. So if p denotes the price of the item and C the total cost of buying the item then if the item is sold at \$ 1 then the cost is $1 + (0.06)(1) = \$1.06$ or $C(1) = \$1.06$. If the item is sold at \$2 then the cost of buying the item is $2 + (0.06)(2) = \$2.12$, or $C(2) = \$2.12$, and so on. Thus we have a relationship between the quantities C and p such that each value of p determines exactly one value of C. In this case, we say that C is a function of p. Describes this function using words, a table, a graph, and a formula.

Solution.
•**Words:** To find the total cost, multiply the price of the item by 0.06 and add the result to the price.
•**Table:** The chart below gives the total cost of buying an item at price p as a function of p for $1 \leq p \leq 6$.

p	1	2	3	4	5	6
C	1.06	2.12	3.18	4.24	5.30	6.36

•**Graph:** The graph of the function C is obtained by plotting the data in the above table. See Figure 1.
•**Formula:** The formula that describes the relationship between C and p is given by

$$C(p) = 1.06p. \blacksquare$$

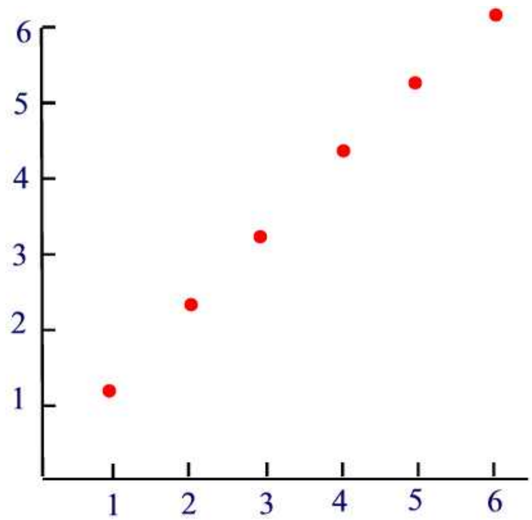


Figure 1

Example 1.2
The income tax T owed in a certain state is a function of the taxable income I, both measured in dollars. The formula is

$$T = 0.11I - 500.$$

(a) Express using functional notation the tax owed on a taxable income of $ 13,000, and then calculate that value.
(b) Explain the meaning of $T(15,000)$ and calculate its value.

Solution.
(a) The functional notation is given by $T(13,000)$ and its value is

$$T(13,000) = 0.11(13,000) - 500 = \$930.$$

(b) $T(15,000)$ is the tax owed on a taxable income of $15,000. Its value is

$$T(15,000) = 0.11(15,000) - 500 = \$1,150. \blacksquare$$

Emphasis of the Four Representations
A formula has the advantage of being both compact and precise. However, not much insight can be gained from a formula as from a table or a graph. A graph provides an overall view of a function and thus makes it easy to deduce important properties. Tables often clearly show trends that are not easily discerned from formulas, and in many cases tables of values are much easier to obtain than a formula.

Remark 1.1
To evaluate a function given by a graph, locate the point of interest on the horizontal axis, move vertically to the graph, and then move horizontally to the vertical axis. The function value is the location on the vertical axis.

Now, most of the functions that we will encounter in this course have formulas. For example, the area A of a circle is a function of its radius r. In function notation, we write $A(r) = \pi r^2$. However, there are functions that can not be represented by a formula. For example, the value of Dow Jones Industrial Average at the close of each business day. In this case the value depends on the date, but there is no known formula. Functions of this nature, are mostly represented by either a graph or a table of numerical data.

Example 1.3
The table below shows the daily low temperature for a one-week period in New York City during July.
(a) What was the low temperature on July 19?
(b) When was the low temperature $73°F$?
(c) Is the daily low temperature a function of the date?Explain.
(d) Can you express T as a formula?

D	17	18	19	20	21	22	23
T	73	77	69	73	75	75	70

Solution.
(a) The low temperature on July 19 was $69°F$.
(b) On July 17 and July 20 the low temperature was $73°F$.
(c) T is a function of D since each value of D determines exactly one value of T.
(d) T can not be expressed by an exact formula.■

So far, we have introduced rules between two quantities that define functions. Unfortunately, it is possible for two quantities to be related and yet for neither quantity to be a function of the other.

Example 1.4
Let x and y be two quantities related by the equation

$$x^2 + y^2 = 4.$$

(a) Is x a function of y? Explain.
(b) Is y a function of x? Explain.

Solution.
(a) For $y = 0$ we have two values of x, namely, $x = -2$ and $x = 2$. So x is not a function of y.
(b) For $x = 0$ we have two values of y, namely, $y = -2$ and $y = 2$. So y is not a function of x. $\blacksquare$

Next, suppose that the graph of a relationship between two quantities x and y is given. To say that y is a function of x means that for each value of x there is exactly one value of y. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same x-value and this violates the definition of a function.

Example 1.5
Which of the graphs (a), (b), (c) in Figure 2 represent y as a function of x?

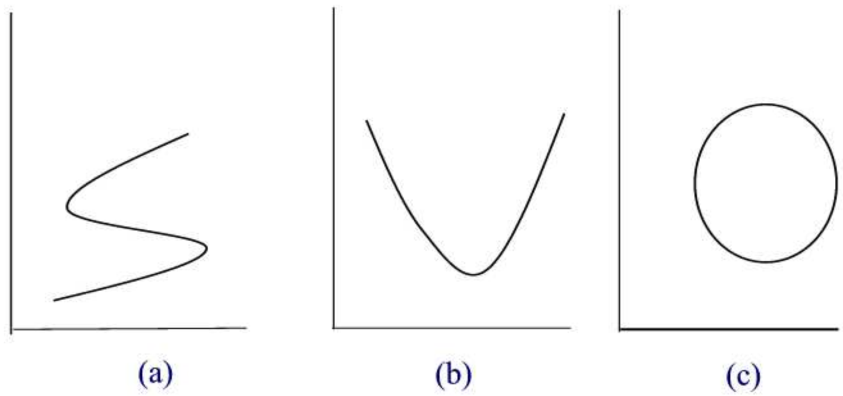


Figure 2

Solution.
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once.■

Recommended Problems (pp. 4 - 6): 1, 3, 5, 10, 13, 15, 17, 24, 25.

2 Linear Functions

This section is designed to introduce students to the concept of linear functions.
A **linear function** f is a function with the property that for any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f, the **difference quotient**

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is constant. We say that y is changing at a constant rate with respect to x. Thus, y changes by the some amount for every unit change in x. Geometrically, the graph is a straight line (and thus the term linear). The constant rate of change, denoted by m, is called the **slope** of the line and Figure 3 shows its geometrical significance.

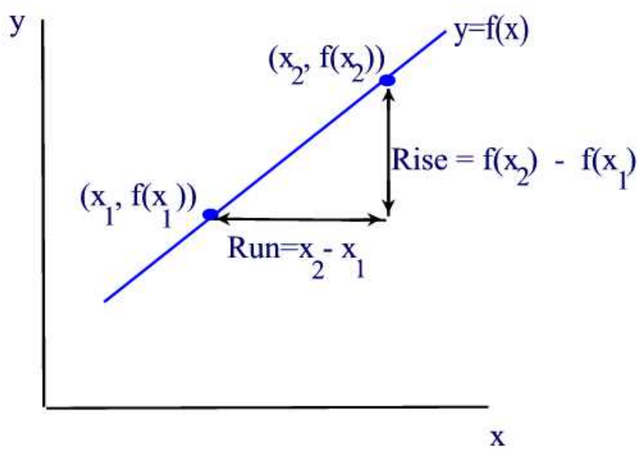


Figure 3

Example 2.1
Show that the function $f(x) = x^2$ is not linear.

Solution.
Taking the points $(0, 0)$ and $(1, 1)$ we find

$$\frac{f(1) - f(0)}{1 - 0} = 1.$$

On the other hand, taking the point $(1,1)$ and $(2,4)$ we find

$$\frac{f(2)-f(1)}{2-1}=3.\blacksquare$$

Thus, f is not linear.$\blacksquare$

Example 2.2
Suppose you pay \$ 192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$ 32. Explain why the function that shows your net income (revenue from sales minus rental fees) as a function of the number of necklaces sold is a linear function.

Solution.
Let $P(n)$ denote the net income from selling n necklaces. Each time a necklace is sold, that is, each time n is increased by 1, the net income P is increased by the same constant, \$32. Thus the rate of change for P is always the same, and hence P is a linear function.$\blacksquare$

Testing Data for Linearity
Next, we will consider the question of recognizing a linear function given by a table.
Let f be a linear function given by a table. Then the rate of change is the same for all pairs of points in the table. In particular, when the x values are evenly spaced the change in y is constant.

Example 2.3
Which of the following tables could represent a linear function?

x	f(x)
0	10
5	20
10	30
15	40

x	g(x)
0	20
10	40
20	50
30	55

Solution.
Since equal increments in x yield equal increments in y then $f(x)$ is a linear function. On the contrary, since $\frac{40-20}{10-0} \neq \frac{50-40}{20-10}$ then $g(x)$ is not linear.$\blacksquare$

It is possible to have a table of linear data in which neither the x-values nor the y-values go up by equal amounts. However, the rate of change of any pairs of points in the table is constant.

Example 2.4
The following table contains linear data, but some data points are missing. Find the missing data points.

x	2	5		8	
y	5		17	23	29

Solution.
Consider the points $(2,5),(5,a),(b,17),(8,23)$, and $(c,29)$. Since the data is linear then we must have $\frac{a-5}{5-2} = \frac{23-5}{8-2}$. That is, $\frac{a-5}{3} = 3$. Cross multiplying to obtain $a-5=9$ or $a=14$. It follows that when x in increased by 1, y increases by 3. Hence, $b=6$ and $c=10$.■

Next, we consider the question of recognizing a linear function defined by an equation.
Linear functions come in three main forms: slope-intercept form, point-slope form, and standard form. Suppose, first, that $f(x)$ is a linear function of x. Then f changes at a constant rate, say m. That is, if we pick two points $(0,f(0))$ and $(x,f(x))$ then

$$m = \frac{f(x)-f(0)}{x-0}.$$

Writing $f(x)$ in terms of x we obtain $f(x) = mx + f(0)$. This is the function notation of the linear function $f(x)$. Another notation is the equation notation, $y = mx + f(0)$. We will denote the number $f(0)$ by b. In this case, the linear function will be written as

$$f(x) = mx + b \quad or \quad y = mx + b.$$

We call this equation the **slope-intercept form** since it involves the slope m and the vertical intercept b.

Example 2.5
The value of a new computer equipment is \$20,000 and the value drops at a constant rate so that it is worth \$ 0 after five years. Let $V(t)$ be the value of the computer equipment t years after the equipment is purchased.

(a) Find the slope m and the y-intercept b.
(b) Find a formula for $V(t)$.

Solution.
(a) Since $V(0) = 20,000$ and $V(5) = 0$ then the slope of $V(t)$ is

$$m = \frac{0 - 20,000}{5 - 0} = -4,000$$

and the vertical intercept is $V(0) = 20,000$.
(b) A formula of $V(t)$ is $V(t) = -4,000t + 20,000$. In financial terms, the function $V(t)$ is known as the **straight-line depreciation** function.■

So far we have represented a linear function by the expression $y = mx + b$. This is known as the **slope-intercept form** of the equation of a line. Now, if the slope m of a line is known and one point (x_0, y_0) is given then by taking any point (x, y) on the line and using the definition of m we find

$$\frac{y - y_0}{x - x_0} = m.$$

Cross multiply to obtain: $y - y_0 = m(x - x_0)$. This is known as the **point-slope form** of a line.

Example 2.6
Find the equation of the line passing through the point $(100, 1)$ and with slope $m = 0.01$.

Solution.
Using the above formula we have: $y - 1 = 0.01(x - 100)$ or $y = 0.01x$.■

Note that the form $y = mx + b$ can be rewritten in the form

$$Ax + By + C = 0. \tag{1}$$

where $A = m, B = -1$, and $C = b$. The form (1) is known as the **standard form** of a linear function.

Example 2.7
Rewrite in standard form: $3x + 2y + 40 = x - y$.

Solution.
Subtracting $x - y$ from both sides to obtain $2x + 3y + 40 = 0$.■

Families of Linear Functions
We have seen that the graph of a linear function $f(x) = mx + b$ is a straight line. But a line can be horizontal, vertical, rising to the right or falling to the right. The slope is the parameter that provides information about the steepness of a straight line.

• If $m = 0$ then $f(x) = b$ is a constant function whose graph is a horizontal line at $(0, b)$.
• For a vertical line, the slope is undefined since any two points on the line have the same x-value and this leads to a division by zero in the formula for the slope. The equation of a vertical line has the form $x = a$.
• Suppose that the line is neither horizontal nor vertical. If $m > 0$ then by Section 3, $f(x)$ is increasing. That is, the line is rising to the right.
• If $m < 0$ then $f(x)$ is decreasing. That is, the line is falling to the right.
• The slope, m, tells us how fast the line is climbing or falling. The larger the value of m the more the line rises and the smaller the value of m the more the line falls.
The parameter b tells us where the line crosses the vertical axis.

Example 2.8
Arrange the slopes of the lines in the figure from largest to smallest.

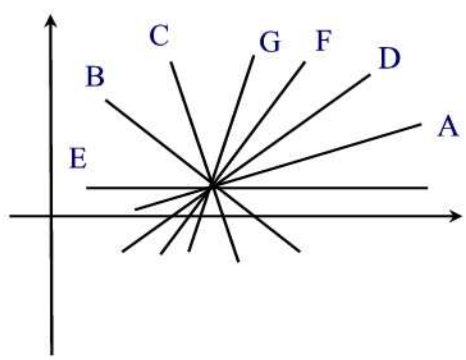


Figure 4

Solution.
According to Figure 4 we have

$$m_G > m_F > m_D > m_A > m_E > m_B > m_C. \blacksquare$$

Recommended Problems (pp. 11 - 13): 1, 3, 7, 9, 11, 12, 15, 17, 23, 25, 29.

3 The Rate of Change

Functions given by tables of values have their limitations in that nearly always leave gaps. One way to fill these gaps is by using the **average rate of change**. For example, Table 1 below gives the population of the United States between the years 1950 - 1990.

d(year)	1950	1960	1970	1980	1990
N(in millions)	151.87	179.98	203.98	227.23	249.40

Table 1

This table does not give the population in 1972. One way to estimate $N(1972)$, is to find the average yearly rate of change of N from 1970 to 1980 given by

$$\frac{227.23 - 203.98}{10} = 2.325 \text{ } \textit{million people per year}.$$

Then,

$$N(1972) = N(1970) + 2(2.325) = 208.63 \text{ } \textit{million}.$$

Average rates of change can be calculated not only for functions given by tables but also for functions given by formulas. The **average rate of change** of a function $y = f(x)$ from $x = a$ to $x = b$ is given by the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{\text{Change in function value}}{\text{Change in x value}} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, this quantity represents the slope of the secant line going through the points $(a, f(a))$ and $(b, f(b))$ on the graph of $f(x)$. See Figure 5. The average rate of change of a function on an interval tells us how much the function changes, on average, per unit change of x within that interval. On some part of the interval, f may be changing rapidly, while on other parts f may be changing slowly. The average rate of change evens out these variations.

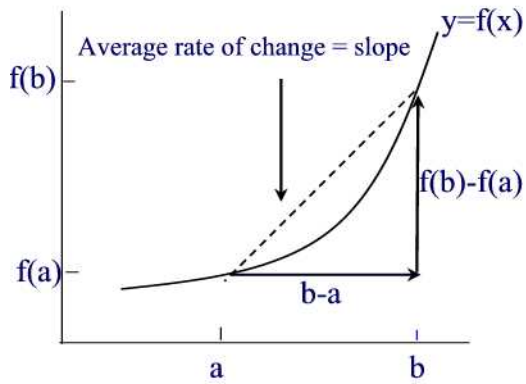


Figure 5

Example 3.1
Find the average value of the function $f(x) = x^2$ from $x = 3$ to $x = 5$.

Solution.
The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(3)}{5 - 3} = \frac{25 - 9}{2} = 8. \blacksquare$$

Example 3.2 *(Average Speed)*
During a typical trip to school, your car will undergo a series of changes in its speed. If you were to inspect the speedometer readings at regular intervals, you would notice that it changes often. The speedometer of a car reveals information about the instantaneous speed of your car; that is, it shows your speed at a particular instant in time. The instantaneous speed of an object is not to be confused with the average speed. Average speed is a measure of the distance traveled in a given period of time. That is,

$$\text{Average Speed} = \frac{\text{Distance traveled}}{\text{Time elapsed}}.$$

If the trip to school takes 0.2 hours (i.e. 12 minutes) and the distance traveled is 5 miles then what is the average speed of your car?

Solution.
The average velocity is given by

$$\text{Ave. Speed} = \frac{5 \text{ miles}}{0.2 \text{ hours}} = 25\text{miles/hour}.$$

This says that on the average, your car was moving with a speed of 25 miles per hour. During your trip, there may have been times that you were stopped and other times that your speedometer was reading 50 miles per hour; yet on the average you were moving with a speed of 25 miles per hour.■

Average Rate of Change and Increasing/Decreasing Functions

Now, we would like to use the concept of the average rate of change to test whether a function is increasing or decreasing on a specific interval. First, we introduce the following definition: We say that a function is **increasing** if its graph climbs as x moves from left to right. That is, the function values increase as x increases. It is said to be **decreasing** if its graph falls as x moves from left to right. This means that the function values decrease as x increases.

As an application of the average rate of change, we can use such quantity to decide whether a function is increasing or decreasing. If a function f is increasing on an interval I then by taking any two points in the interval I, say $a < b$, we see that $f(a) < f(b)$ and in this case

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Going backward with this argument we see that if the average rate of change is positive in an interval then the function is increasing in that interval. Similarly, if the average rate of change is negative in an interval I then the function is decreasing there.

Example 3.3

The table below gives values of a function $w = f(t)$. Is this function increasing or decreasing?

t	0	4	8	12	16	20	24
w	100	58	32	24	20	18	17

Solution.

The average of w over the interval $[0, 4]$ is

$$\frac{w(4) - w(0)}{4 - 0} = \frac{58 - 100}{4 - 0} = -10.5$$

The average rate of change of the remaining intervals are given in the chart below

time interval	[0,4]	[4,8]	[8,12]	[12,16]	[16, 20]	[20,24]
Average	-10.5	-6.5	-2	-1	-0.5	-0.25

Since the average rate of change is always negative on $[0, 24]$ then the function is decreasing on that interval. Of Course, you can see from the table that the function is decreasing since the output values are decreasing as x increases. The purpose of this problem is to show you how the average rate of change is used to determine whether a function is increasing or decreasing.■

Some functions can be increasing on some intervals and decreasing on other intervals. These intervals can often be identified from the graph.

Example 3.4
Determine the intervals where the function is increasing and decreasing.

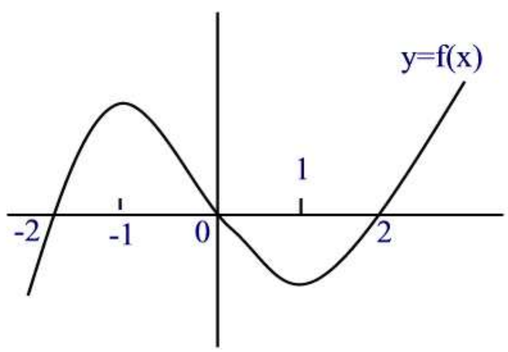


Figure 6

Solution.
The function is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on the interval $(-1, 1)$.■

Rate of Change and Concavity
We have seen that when the rate of change of a function is constant then its graph is a straight line. However, not all graphs are straight lines; they may bend up or down as shown in the following two examples.

Example 3.5
Consider the following two graphs in Figure 7.

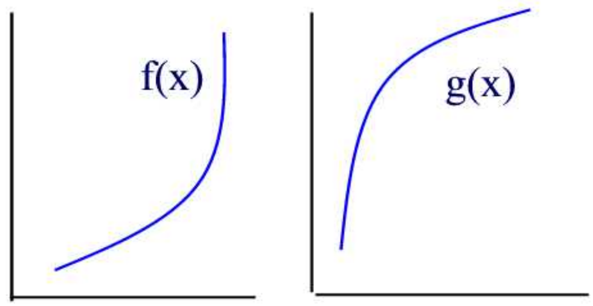


Figure 7

(a) What do the graphs above have in common?
(b) How are they different? Specifically, look at the rate of change of each.

Solution.
(a) Both graphs represent increasing functions.
(b) The rate of change of $f(x)$ is more and more positive so the graph bends up whereas the rate of change of $g(x)$ is less and less positive and so it bends down. ■

The following example deals with version of the previous example for decreasing functions.

Example 3.6
Consider the following two graphs given in Figure 8.

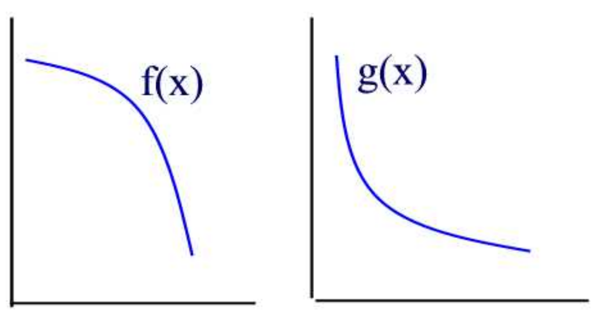


Figure 8

(a) What do the graphs above have in common?
(b) How are they different? Specifically, look at the rate of change of each.

Solution.
(a) Both functions are decreasing.
(b) The rate of change of $f(x)$ is more and more negative so the graph bends down, whereas the rate of change of $g(x)$ is less and less negative so the graph bends up.

Conclusions:
• When the rate of change of a function is increasing then the function is **concave up.** That is, the graph bends upward.
• When the rate of change of a function is decreasing then the function is **concave down**. That is, the graph bends downward.

The following example discusses the concavity of a function given by a table.

Example 3.7
Given below is the table for the function $H(x)$. Calculate the rate of change for successive pairs of points. Decide whether you expect the graph of $H(x)$ to concave up or concave down?

x	*12*	*15*	*18*	*21*
H(x)	*21.40*	*21.53*	*21.75*	*22.02*

Solution.

$$\begin{aligned}
\frac{H(15)-H(12)}{15-12} &= \frac{21.53-21.40}{3} \approx 0.043\\
\frac{H(18)-H(15)}{18-15} &= \frac{21.75-21.53}{3} \approx 0.073\\
\frac{H(21)-H(18)}{21-18} &= \frac{22.02-21.75}{3} \approx 0.09
\end{aligned}$$

Since the rate of change of $H(x)$ is increasing then the function is concave up.■

Remark 3.1
Since the graph of a linear function is a straight line, that is its rate of change is constant, then it is neither concave up nor concave down.

Recommended Problems (pp. 19 - 21): 1, 2, 3, 4, 5, 7, 10, 12, 13, 16, 17, 25, 27, 30.

4 Applications of Functions to Economics

The goal of this section is to exhibit some functions used in business and economics.

The **cost function** C gives the cost $C(q)$ of manufacturing a quantity q of some good. A **linear cost function** has the form

$$C(q) = mq + b,$$

where the y-intercept b is called the **fixed cost**, i.e. the costs incurred even if nothing is produced, and the slope m is called the **variable costs per unit**.
The function $\overline{C(x)} = \frac{C(x)}{x}$ is called the **average cost function.**

Example 4.1
What is the cost function of manufacturing a product with fixed cost of $400 and variable costs of $40 per item, assuming the function is linear?

Solution.
The cost function is

$$C(q) = 40q + 400. \blacksquare$$

Example 4.2
Values of a linear cost function are shown below. What are the fixed costs and the variable costs per units? Find a formula for the cost function.

q	0	5	10	15	20
C(q)	5000	5020	5040	5060	5080

Solution.
The fixed costs are $b = C(0) = \$5,000$, the variable costs are

$$m = \frac{5020 - 5000}{5 - 0} = 4$$

The cost function is

$$C(q) = 4q + 5,000 \blacksquare$$

A **revenue function** R gives the total revenue $R(q)$ from the sale of a quantity q at a unit price p dollars. Thus, $R(q) = pq$.

Example 4.3
A company that makes a certain brand of chairs has fixed costs of $\$5,000$ and marginal cost of $\$30$ per chair. The company sells the chairs for $\$50$ each. Find formulas for the cost and revenue functions.

Solution.
The cost function is $C(q) = 30q + 5000$. The revenue function is $R(q) = pq = 30q$.$\blacksquare$

In any business, decisions are made based on the profit function. **Profit** is defined to be revenue minus cost. That is

$$P(q) = R(q) - C(q).$$

The **break-even point** is the point where the profit is zero, i.e. $R(q) = C(q)$.

Example 4.4
A company has cost and revenue functions, in dollars, given by $C(q) = 6,000 + 10q$ and $R(q) = 12q$.

(a) Graph the functions $C(q)$ and $R(q)$ on the same coordinate axes.
(b) Find the break-even point and illustrate it graphically.
(c) When does the company make a profit? Loses money?

Solution.
(a) The graph is given in Figure 9.

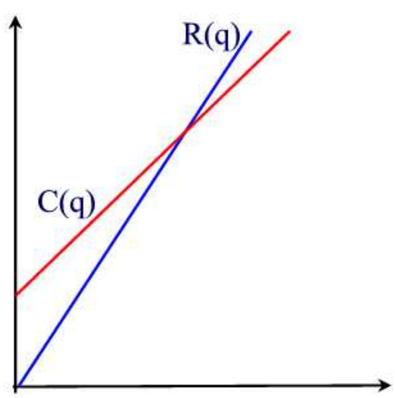


Figure 9

(b) The break-even point is the point of intersection of the two lines. To find the point, set $12q = 10q + 6000$ and solve for q to find $q = 3000$. Thus, the break-even point is the point $(3000, 36000)$.
(c) The company makes profit for $q > 3000$ and loses money for $q < 3000$.■

Marginal Analysis
In economics and business the term marginal stands for a rate of change. Marginal analysis is an area of economics concerned with estimating the effect on quantities such as cost, revenue, and profit when the level of production is changed by a unit amount. For example, if $C(q)$ is the cost of producing q units of a certain commodity, then the **marginal cost**, MC(q), is the additional cost of producing one more unit and is given by the difference $MC(q) = C(q + 1) - C(q)$.

Example 4.5
Let $C(q)$ represent the cost, $R(q)$ the revenue, and $P(q)$ the total profit, in dollars, of producing q units.

(a) If $MC(50) = 75$ and $MR(50) = 84$, approximately how much profit is earned by the 51st item?
(b) If $MC(90) = 71$ and $MR(90) = 68$, approximately how much profit is earned by the 91st item?

Solution.
(a) $MP(50) = MR(50) - MC(50) = 84 - 75 = 9$.
(b) $MP(90) = MR(90) - MC(90) = 68 - 71 = -3$. A loss by 3 dollars.■

Remark 4.1
Marginal cost and average cost can differ greatly. For example, suppose it costs \$1000 to produce 100 units and \$1020 to produce 101 units. The average cost per unit is \$10, but the marginal cost of the 101st unit is \$20. Similar remarks apply for the marginal revenue and the marginal profit.

The Depreciation Function
An important application of linear functions in financial modeling is the depreciation function.
In a financial setting, a linear function with negative slope is called a **depreciation function.**

Example 4.6
A new sports car costs \$40,000 and depreciates \$3000 per year.

(a) Determine an equation for the depreciation function.
(b) How much will the car be worth in 5 years?

Solution.
Since the rate of depreciation is constant then the depreciation function is a linear function, say, $V(t) = mt + b$. Since $b = V(0)$ then $b = \$40,000$. Also, $m = -\$3000$ per year. Thus, $V(t) = -3000t + 40,000$.
(b) The question here is equivalent to finding $V(5)$. That is, $V(5) = -3000(5) + 40,000 = \$25,000$.■

Supply and Demand Curves
The quantity q manufactured and sold depends on the unit price p. In general, when the price goes up then manufacturers are willing to supply more of the product whereas consumers are going to reduce their buyings. Since consumers and manufacturers react differently to changes in price, there are two curves relating p and q.

The **supply curve** is the quantity that producers are willing to make at a given price. Thus, increasing price will increase quantity.
The **demand curve** is the amount that will be bought by consumers at a given price. Thus, decreasing price will increase quantity.
Even though quantity is a function of price, it is the tradition to use the vertical y-axis for the variable p and the horizontal x-axis for the variable q. The supply and demand curves intersect at point (q^*, p^*) called the **point of equilibrium**. We call p^* the **equilibrium price** and q^* the **equilibrium quantity**.

Example 4.7
Find the equilibrium point for the supply function $S(p) = 3p - 50$ and the demand function $D(p) = 100 - 2p$.

Solution.
Setting the equation $S(p^*) = D(p^*)$ to obtain $3p^* - 50 = 100 - 2p^*$. By adding $2p^* + 50$ to both sides we obtain $5p^* = 150$. Solving for p^* we find $p^* = 30$. Substituting this value in $S(p)$ we find $q^* = 3(30) - 50 = 40$.■

The impact of Taxes on Equilibrium

Now let us consider the previous problem again. Suppose that the government imposes a \$ 5 tax per item on the supplier. How does this increase affect the equilibrium price p? By imposing the \$ 5 tax per item the new quantity to be supplied is now given by

$$S(p-5) = 3(p-5) - 50 = 3p - 65.$$

However, the demand function is the same. Calculating the equilibrium price we find

$$\begin{aligned} 3p - 65 &= 100 - 2p \\ 5p &= 165 \\ p &= \$33. \end{aligned}$$

Thus, the previous equilibrium price has increased by \$ 3. This means that the consumer's share of the \$5 tax is \$3 whereas the supplier share is \$2. That is, even though the tax was imposed on the producer, some of the tax is passed on to the consumer in terms of higher prices.

Recommended Problems (pp. 29 - 32): 1, 3, 4, 5, 7, 8, 11, 14, 18, 20, 24, 26.

5 Exponential Functions

Exponential functions appear in many applications such as population growth, radioactive decay, and interest on bank loans.
Recall that linear functions are functions that change at a constant rate. For example, if $f(x) = mx + b$ then $f(x+1) = m(x+1) + b = f(x) + m$. So when x increases by 1, the y value increases by m. In contrast, an exponential function with base a is one that changes by constant multiples of a. That is, $f(x+1) = af(x)$. Writing $a = 1 + r$ we obtain $f(x+1) = f(x) + rf(x)$. Thus, an exponential function is a function that changes at a constant percent rate.
Exponential functions are used to model increasing quantities such as **population growth** problems.

Example 5.1
Suppose that you are observing the behavior of cell duplication in a lab. In one experiment, you started with one cell and the cells doubled every minute. That is, the population cell is increasing at the constant rate of 100%. Write an equation to determine the number (population) of cells after one hour.

Solution.
Table 2 below shows the number of cells for the first 5 minutes. Let $P(t)$ be the number of cells after t minutes.

t	0	1	2	3	4	5
P(t)	1	2	4	8	16	32

Table 2

At time 0, i.e t=0, the number of cells is 1 or $2^0 = 1$. After 1 minute, when $t = 1$, there are two cells or $2^1 = 2$. After 2 minutes, when $t = 2$, there are 4 cells or $2^2 = 4$.
Therefore, one formula to estimate the number of cells (size of population) after t minutes is the equation (model)

$$f(t) = 2^t.$$

It follows that $f(t)$ is an increasing function. Computing the rates of change to obtain

$$\begin{aligned}
\tfrac{f(1)-f(0)}{1-0} &= 1\\
\tfrac{f(2)-f(1)}{2-1} &= 2\\
\tfrac{f(3)-f(2)}{3-2} &= 4\\
\tfrac{f(4)-f(3)}{4-3} &= 8\\
\tfrac{f(5)-f(4)}{5-4} &= 16.
\end{aligned}$$

Thus, the rate of change is increasing. Geometrically, this means that the graph of $f(t)$ is concave up. See Figure 10.

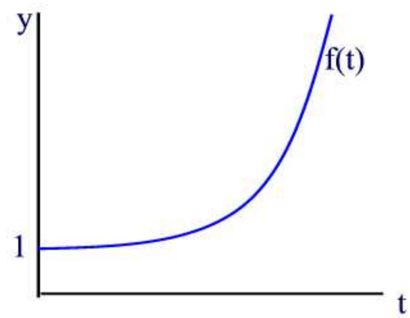


Figure 10

Now, to determine the number of cells after one hour we convert to minutes to obtain $t = 60$ minutes so that $f(60) = 2^{60} = 1.15 \times 10^{18}$ cells. ■

Exponential functions can also model decreasing quantities known as **decay models.**

Example 5.2
If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 50,000 cells?

Solution.
Let $P(t)$ be the number of cells after t minutes. Then $P(t+1) = P(t) - 45\%P(t)$ or $P(t+1) = 0.55P(t)$. By constructing a table of data we find

t	P(t)
0	5,000,000
1	2,750,000
2	1,512,500
3	831,875
4	457,531.25
5	251,642.19
6	138,403.20
7	76,121.76
8	41,866.97

So it takes 8 minutes for the population to reduce to less than 50,000 cells. A formula of $P(t)$ is $P(t) = 5,000,000(0.55)^t$. The graph of $P(t)$ is given in Figure 11.■

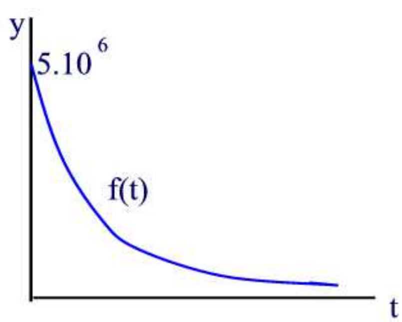


Figure 11

From the previous two examples, we see that an exponential function has the general form

$$P(t) = b \cdot a^t, a > 0 \ a \neq 1.$$

Since $b = P(0)$ then we call b the **initial value.** We call a the base of $P(t)$. If $a > 1$, then $P(t)$ shows exponential growth with **growth factor** a. The graph of P will be similar in shape to that in Figure 10.
If $0 < a < 1$, then P shows exponential decay with **decay factor** a. The graph of P will be similar in shape to that in Figure 11.
Since $P(t+1) = aP(t)$ then $P(t+1) = P(t) + rP(t)$ where $r = a - 1$. We call r the **percent growth rate**.

Remark 5.1
Why a is restricted to $a > 0$ and $a \neq 1$? Since t is allowed to have any value then a negative a will create meaningless expressions such as $\sqrt{a}$ (if $t = \frac{1}{2}$). Also, for $a = 1$ the function $P(t) = b$ is called a **constant function** and its graph is a horizontal line.

Example 5.3
Suppose you are offered a job at a starting salary of \$40,000 per year. To strengthen the offer, the company promises annual raises of 6% per year for the first 10 years. Let $P(t)$ be your salary after t years. Find a formula for $P(t)$ and then compute your projected salary after 4 years from now.

Solution.
A formula of $P(t)$ is $P(t) = 40,000(1.06)^t$. After four years, the projected salary is $P(3) = 40,000(1.06)^4 \approx 50,499.08$.■

Example 5.4
The amount in milligrams of a drug in the body t hours after taking a pill is given by $A(t) = 25(0.85)^t$.
(a) What is the initial dose given?
(b) What percent of the drug leaves the body each hour?
(c) What is the amount of drug left after 10 hours?

Solution.
(a) Initial dose give is $A(0) = 25$ mg.
(b) $r = a - 1 = 0.85 - 1 = -.15$ so that 15% of the drug leaves the body each hour.
(c) $A(10) = 25(0.85)^{10} \approx 4.92$ mg.■

Recognizing an Exponential Function Defined by Data
Suppose that f is a function defined by a table of values. If f is an exponential function then f can be written in the form $f(x) = ba^x$. Thus, $\frac{f(x+n)}{f(x)} = \frac{ba^{x+n}}{bax^n} = a^n$. This says that the ratios of y values are constant for equally spaced x values.

Example 5.5
Decide if the function is linear or exponential?Find a formula for each case.

x	0	1	2	3	4
f(x)	12.5	13.75	15.125	16.638	18.301
g(x)	0	2	4	6	8

Solution.
Since $\frac{13.75}{12.5} \approx \frac{15.125}{13.75} \approx \frac{16.638}{15.125} \approx \frac{18.301}{16.638} \approx 1.1$ then $f(x)$ is an exponential function.
To find a formula for $f(x) = ba^x$ we use the first two points obtaining $12.5 = f(0) = b$ and $13.75 = f(1) = ba = 12.5a$. Hence, $a = \frac{13.75}{12.5} \approx 1.1$ so that $f(x) = 12.5(1.1)^x$.
On the other hand, equal increments in x corresponds to equal increments in the g-values so that $g(x)$ is linear, say $g(x) = mx + b$. Since $g(0) = 0$ then $b = 0$. Also, $2 = g(1) = m$ so that $g(x) = 2x$. ■

The Effect of the Parameters a and b
Recall that an exponential function with base a and initial value b is a function of the form $f(x) = b \cdot a^x$. In what follows, we assume that $b > 0$. Since $b = f(0)$ then $(0, b)$ is the vertical intercept of $f(x)$.
Let's see the effect of the parameter b on the graph of $f(x) = ba^x$.

Example 5.6
Graph, on the same axes, the exponential functions $f_1(x) = 2 \cdot (1.1)^x$, $f_2(x) = (1.1)^x$, and $f_3(x) = 0.75(1.1)^x$.

Solution.
The three functions as shown in Figure 12.

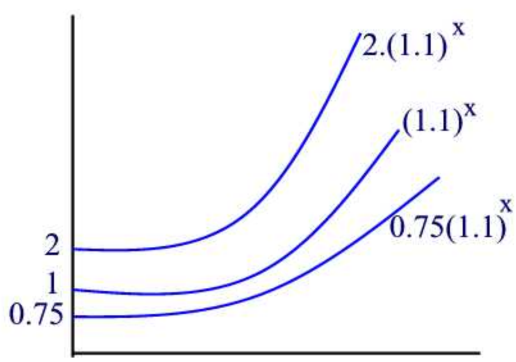


Figure 12

Note that these functions have the same growth factor but different b and therefore different vertical intercepts. ■

We know that the slope of a linear function measures the steepness of the graph. Similarly, the parameter a measures the steepness of the graph of an exponential function. First, we consider the effect of the growth factor on the graph.

Example 5.7
Graph, on the same axes, the exponential functions $f_1(x) = 4^x, f_2(x) = 3^x$, and $f_3(x) = 2^x$.

Solution.
Using a graphing calculator we find

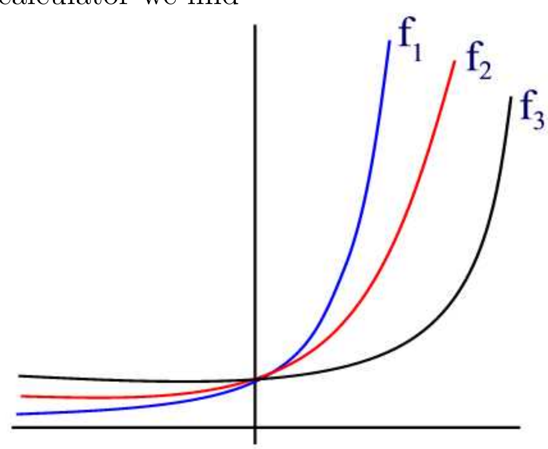


Figure 13

It follows that the greater the value of a, the more rapidly the graph rises. That is, the growth factor a affects the steepness of an exponential function. Also note that as x decreases, the function values approach the x-axis. Symbolically, as $x \to -\infty, y \to 0$.■

Next, we study the effect of the decay factor on the graph.

Example 5.8
Graph, on the same axes, the exponential functions $f_1(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$, $f_2(x) = 3^{-x}$, and $f_3(x) = 4^{-x}$.

Solution.
Using a graphing calculator we find

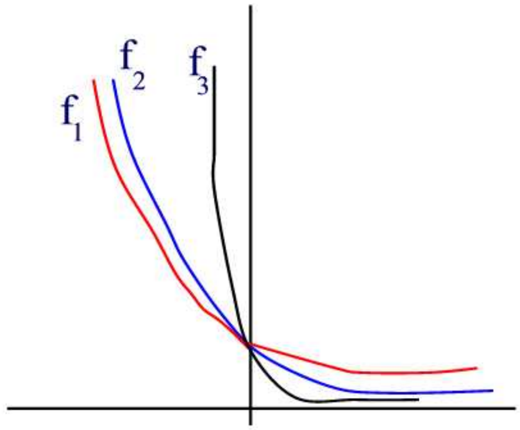


Figure 14

It follows that the smaller the value of a, the more rapidly the graph falls. Also as x increases, the function values approach the x-axis. Symbolically, as $x \to \infty, y \to 0$.

• **General Observations**
(i) For $a > 1$, as x decreases, the function values get closer and closer to 0. Symbolically, as $x \to -\infty, y \to 0$. For $0 < a < 1$, as x increases, the function values gets closer and closer to the x-axis. That is, as $x \to \infty, y \to 0$. We call the x-axis, a **horizontal asymptote**.
(ii) The domain of an exponential function consists of the set of all real numbers whereas the range consists of the set of all positive real numbers.
(iii) The graph of $f(x) = ba^x$ with $b > 0$ is always concave up.

Recommended Problems (pp. 37 - 9): 1, 3, 7, 8, 10, 11, 13, 15, 20, 25.

6 Logarithms and their Properties

An equation of the form $a^x = b$ can be solved graphically. That is, using a calculator we graph the horizontal line $y = b$ and the exponential function $y = a^x$ and then find the point of intersection.
In this section we discuss an algebraic way to solve equations of the form $a^x = b$ where a and b are positive constants. For this, we introduce a function that is found in today's calculators, namely, the function $\ln x$.

$$y = \ln x \quad if \ and \ only \ if \quad e^y = x.$$

where $e = 2.71828\cdots$ We call $\ln x$ the **natural logarithm** of x.

Properties of Logarithms

(i) Since $e^x = e^x$ we can write

$$\ln e^x = x$$

(ii) Since $\ln x = \ln x$ then

$$e^{\ln x} = x$$

(iii) $\ln 1 = 0$ since $e^0 = 1$.
(iv) $\ln e = 1$ since $e^1 = e$.
(v) Suppose that $m = \ln a$ and $n = \ln b$. Then $a = e^m$ and $b = e^n$. Thus, $a \cdot b = e^m \cdot e^n = e^{m+n}$. Rewriting this using logs instead of exponents, we see that

$$\ln (a \cdot b) = m + n = \ln a + \ln b.$$

(vi) If, in (v), instead of multiplying we divide, that is $\frac{a}{b} = \frac{e^m}{e^n} = e^{m-n}$ then using logs again we find

$$\ln \left(\frac{a}{b}\right) = \ln a - \ln b.$$

(vii) It follows from (vi) that if $a = b$ then $\ln a - \ln b = \ln 1 = 0$ that is $\ln a = \ln b$.
(viii) Now, if $n = \ln b$ then $b = e^n$. Taking both sides to the power k we find $b^k = (e^n)^k = e^{nk}$. Using logs instead of exponents we see that $\ln b^k = nk = k \ln b$ that is

$$\ln b^k = k \ln b.$$

Example 6.1
Solve the equation: $4(1.171)^x = 7(1.088)^x$.

Solution.
Rewriting the equation into the form $\left(\frac{1.171}{1.088}\right)^x = \frac{7}{4}$ and then using properties (vii) and (viii) to obtain

$$x \ln\left(\frac{1.171}{1.088}\right) = \ln\frac{7}{4}.$$

Thus,

$$x = \frac{\log\frac{7}{4}}{\ln\left(\frac{1.171}{1.088}\right)}.\blacksquare$$

Example 6.2
Solve the equation $\ln(2x+1) + 3 = 0$.

Solution.
Subtract 3 from both sides to obtain $\ln(2x+1) = -3$. Switch to exponential form to get $2x + 1 = e^{-3}$. Subtract 1 and then divide by 2 to obtain $x = -0.4995$. $\blacksquare$

Remark 6.1
Keep in mind the following:
$\ln(a+b) \neq \ln a + \ln b$. For example, $\ln 2 \neq \ln 1 + \ln 1 = 0$.
$\ln(a-b) \neq \ln a - \ln b$. For example, $\ln(2-1) = \ln 1 = 0$ whereas $\ln 2 - \ln 1 = \ln 2 \neq 0$.
$\ln(ab) \neq \ln a \cdot \ln b$. For example, $\ln 1 = \ln(2 \cdot \frac{1}{2}) = 0$ whereas $\ln 2 \cdot \ln\frac{1}{2} = -\ln^2 2 \neq 0$.
$\ln\left(\frac{a}{b}\right) \neq \frac{\ln a}{\ln b}$. For example, letting $a = b = 2$ we find that $\ln\frac{a}{b} = \ln 1 = 0$ whereas $\frac{\ln a}{\ln b} = 1$.
$\ln\left(\frac{1}{a}\right) \neq \frac{1}{\ln a}$. For example, $\ln\frac{1}{\frac{1}{2}} = \ln 2$ whereas $\frac{1}{\ln\frac{1}{2}} = -\frac{1}{\ln 2}$.

Example 6.3
Sketch the graphs of the functions $y = \ln x$ and $y = e^x$ on the same axes.

Solution.
The graphs of $y = \ln x$ and $y = e^x$ are reflections of one another across the line $y = x$ as shown in Figure 15.■

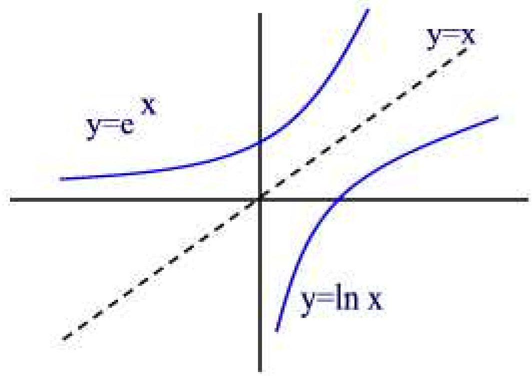


Figure 15

Continuous Growth Rate and the Number e
When writing $y = be^t$ then we say that y is an exponential function with base e. Suppose that $Q(t) = ba^t$. Then $a = e^k$ where $k = \ln a$. Thus,

$$Q(t) = b(e^k)^t = be^{kt}.$$

Note that if $k > 0$ then $e^k > 1$ so that $Q(t)$ represents an exponential growth and if $k < 0$ then $e^k < 1$ so that $Q(t)$ is an exponential decay.
We call the constant k the **continuous growth rate**.

Example 6.4
If $f(t) = 3(1.072)^t$ is rewritten as $f(t) = 3e^{kt}$, find k.

Solution.
By comparison of the two functions we find $e^k = 1.072$. Solving this equation we find $k = \ln 1.072 \approx 0.695$.■

Example 6.5
A population increases from its initial level of 7.3 million at the continuous rate of 2.2% per year. Find a formula for the population $P(t)$ as a function of the year t. When does the population reach 10 million?

Solution.
We are given the initial value 7.3 million and the continuous growth rate $k = 0.022$. Therefore, $P(t) = 7.3e^{0.022t}$. Next,we want to find the time when $P(t) = 10$. That is , $7.3e^{0.022t} = 10$. Divide both sides by 7.3 to obtain $e^{0.022t} \approx 1.37$. Solving this equation to obtain $t = \frac{ln 1.37}{0.022} \approx 14.3.$■

Next, in order to convert from $Q(t) = be^{kt}$ to $Q(t) = ba^t$ we let $a = e^k$. For example, to convert the formula $Q(t) = 7e^{0.3t}$ to the form $Q(t) = ba^t$ we let $b = 7$ and $a = e^{0.3} \approx 1.35$. Thus, $Q(t) = 7(1.35)^t$.

Example 6.6
Find the annual percent rate and the continuous percent growth rate of $Q(t) = 200(0.886)^t$.

Solution.
The annual percent of decrease is $r = a - 1 = 0.886 - 1 = -0.114 = -11.4\%$. To find the continuous percent growth rate we let $e^k = 0.886$ and solve for k to obtain $k = \ln 0.886 \approx -0.121 = -12.1\%.$■

Recommended Problems (pp. 42 - 3): 3, 5, 9, 11, 15, 17, 19, 22, 23, 26, 27, 31.

7 Exponential Growth and Decay

In this section, we consider some applications of exponential functions.

Doubling Time
In some exponential models one is interested in finding the time for an exponential growing quantity to double. We call this time the **doubling time**. To find it, we start with the equation $b \cdot a^t = 2b$ or $a^t = 2$. Solving for t we find $t = \frac{\ln 2}{\ln a}$.

Example 7.1
Find the doubling time of a population growing according to $P = P_0 e^{0.2t}$.

Solution.
Setting the equation $P_0 e^{0.2t} = 2P_0$ and dividing both sides by P_0 to obtain $e^{0.2t} = 2$. Take ln of both sides to obtain $0.2t = \ln 2$. Thus, $t = \frac{\ln 2}{0.2} \approx 3.47.$■

Half-Life
On the other hand, if a quantity is decaying exponentially then the time required for the quantity to reduce into half is called the **half-life**. To find it, we start with the equation $ba^t = \frac{b}{2}$ and we divide both sides by b to obtain $a^t = 0.5$. Take the log of both sides to obtain $t \ln a = \ln(0.5)$. Solving for t we find $t = \frac{\ln(0.5)}{\ln a}$.

Example 7.2
The half-life of Iodine-123 is about 13 hours. You begin with 50 grams of this substance. What is a formula for the amount of Iodine-123 remaining after t hours?

Solution.
Since the problem involves exponential decay then if $Q(t)$ is the quantity remaining after t hours then $Q(t) = 50a^t$ with $0 < t < 1$. But $Q(13) = 25$. That is, $50a^{13} = 25$ or $a^{13} = 0.5$. Thus $a = (0.5)^{\frac{1}{13}} \approx 0.95$ and $Q(t) = 50(0.95)^t$.■

Compound Interest
The term **compound interest** refers to a procedure for computing interest whereby the interest for a specified interest period is added to the original principal. The resulting sum becomes a new principal for the next interest

period. The interest earned in the earlier interest periods earn interest in the future interest periods.
Suppose that you deposit P dollars into a saving account that pays annual interest r and the bank agrees to pay the interest at the end of each time period(usually expressed as a fraction of a year). If the number of periods in a year is n then we say that the interest is **compounded** n times per year (e.g.,'yearly'=1, 'quarterly'=4, 'monthly'=12, etc.). Thus, at the end of the first period the balance will be

$$B = P + \frac{r}{n}P = P\left(1 + \frac{r}{n}\right).$$

At the end of the second period the balance is given by

$$B = P\left(1 + \frac{r}{n}\right) + \frac{r}{n}P\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2.$$

Continuing in this fashion, we find that the balance at the end of the first year, i.e. after n periods, is

$$B = P\left(1 + \frac{r}{n}\right)^n.$$

If the investment extends to another year than the balance would be given by

$$P\left(1 + \frac{r}{n}\right)^{2n}.$$

For an investment of t years then balance is given by

$$B = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Since $\left(1 + \frac{r}{n}\right)^{nt} = \left[\left(1 + \frac{r}{n}\right)^n\right]^t$ then the function B can be written in the form $B(t) = Pa^t$ where $a = \left(1 + \frac{r}{n}\right)^n$. That is, B is an exponential function.

Remark 7.1
Interest given by banks are known as **nominal rate** (e.g. "in name only"). When interest is compounded more frequently than once a year, the account effectively earns more than the nominal rate. Thus, we distinguish between nominal rate and **effective rate.** The effective annual rate tells how much interest the investment actually earns. The quantity $(1 + \frac{r}{n})^n - 1$ is known as the **effective interest rate**.

Example 7.3
Translating a value to the future is referred to as **compounding**. What will be the maturity value of an investment of \$15,000 invested for four years at 9.5% compounded semi-annually?

Solution.
Using the formula for compound interest with $P = \$15,000, t = 4, n = 2$, and $r = .095$ we obtain

$$B = 15,000\left(1 + \frac{0.095}{2}\right)^8 \approx \$21,743.20 \blacksquare$$

Example 7.4
Translating a value to the present is referred to as **discounting.** We call $(1 + \frac{r}{n})^{-nt}$ the **discount factor**. What principal invested today will amount to \$8,000 in 4 years if it is invested at 8% compounded quarterly?

Solution.
The present value is found using the formula

$$P = B\left(1 + \frac{r}{n}\right)^{-nt} = 8,000\left(1 + \frac{0.08}{4}\right)^{-16} \approx \$5,827.57 \blacksquare$$

Example 7.5
What is the effective rate of interest corresponding to a nominal interest rate of 5% compounded quarterly?

Solution.

$$\text{effective rate} = \left(1 + \frac{0.05}{4}\right)^4 - 1 \approx 0.051 = 5.1\% \blacksquare$$

Continuous Compound Interest
When the compound formula is used over smaller time periods the interest becomes slightly larger and larger. That is, frequent compounding earns a higher effective rate, though the increase is small.
This suggests compounding more and more, or equivalently, finding the value of B in the long run. In Calculus, it can be shown that the expression $\left(1 + \frac{r}{n}\right)^n$ approaches e^r as $n \to \infty$, where e (named after Euler) is a number whose value is $e = 2.71828\cdots$. The balance formula reduces to $B = Pe^{rt}$. This formula is known as the **continuous compound formula**. In this case, the annual effective interest rate is found using the formula $e^r - 1$.

Example 7.6
Find the effective rate if \$1000 is deposited at 5% annual interest rate compounded continuously.

Solution.
The effective interest rate is $e^{0.05} - 1 \approx 0.05127 = 5.127\%$■

Example 7.7
Which is better: An account that pays 8% annual interest rate compounded quarterly or an account that pays 7.95% compounded continuously?

Solution.
The effective rate corresponding to the first option is

$$\left(1 + \frac{0.08}{4}\right)^4 - 1 \approx 8.24\%$$

That of the second option

$$e^{0.0795} - 1 \approx 8.27\%$$

Thus, we see that 7.95% compounded continuously is better than 8% compounded quarterly. ■

Present and Future Value
Many business deals involve payments in the future. For example, when a car or a home is bought on credits, payments are made over a period of time. The **future value**, FV, of a payment P is the amount to which P would have grown if deposited today in an interest bearing bank account. The **present value**, PV, of a future payment FV, is the amount that would have to be deposited in a bank account today to produce exactly FV in the account at the relevant time future.
If interest is compounded n times a year at a rate r for t years, then the relationship between FV and PV is given by the formula

$$FV = PV(1 + \frac{r}{n})^{nt}.$$

In the case of continuous compound interest, the forumla is given by

$$FV = PVe^{rt}.$$

Example 7.8
You need \$10,000 in your account 3 years from now and the interest rate is 8% per year, compounded continuously. How much should you deposit now?

Solution.
We have $FV = \$10,000, r = 0.08, t = 3$ and we want to find PV. Solving the formula $FV = PVe^{rt}$ for PV we find $PV = FVe^{-rt}$. Substituting to obtain, $PV = 10,000e^{-0.24} \approx \$7,866.28$.■

Recommended Problems (pp. 48 - 51): 5, 7, 8, 13, 19, 22, 28, 29, 30, 33.

8 Building New Functions from Old Ones

In this section we discuss various ways for building new functions from old ones. New functions can be obtained by composing functions, using arithmetic combinations, and finally by making changes to either the input or the output of a function.

Composition of Functions
The first procedure for building new functions from old ones known is the composition of functions.
We start with an example of a real-life situation where composite functions are applied.

Example 8.1
You have two money machines, both of which increase any money inserted into them. The first machine doubles your money. The second adds five dollars. The money that comes out is described by $f(x) = 2x$ in the first case, and $g(x) = x + 5$ in the second case, where x is the number of dollars inserted. The machines can be hooked up so that the money coming out of one machine goes into the other. Find formulas for each of the two possible composition machines.

Solution.
Suppose first that x dollars enters the first machine. Then the amount of money that comes out is $f(x) = 2x$. This amount enters the second machine. The final amount coming out is given by $g(f(x)) = f(x) + 5 = 2x + 5$.
Now, if x dollars enters the second machine first, then the amount that comes out is $g(x) = x + 5$. If this amount enters the second machine then the final amount coming out is $f(g(x)) = 2(x + 5) = 2x + 10.\blacksquare$
The function $f(g(x))$ is called the **composition** of the functions f and g; the function $g(f(x))$ is called the **composition** of the functions g and f.

In general, suppose we are given two functions f and g such that the range of g is contained in the domain of f so that the output of g can be used as input for f. We define a new function, called the **composition** of f and g, by the formula $f(g(x))$ where g is the inside function and f is the outside function. In a similar way, we can define the composition of g and f to be the function

$$g(f(x))$$

so that the output of f is the input of g.
Using a Venn diagram (See Figure 16) we have

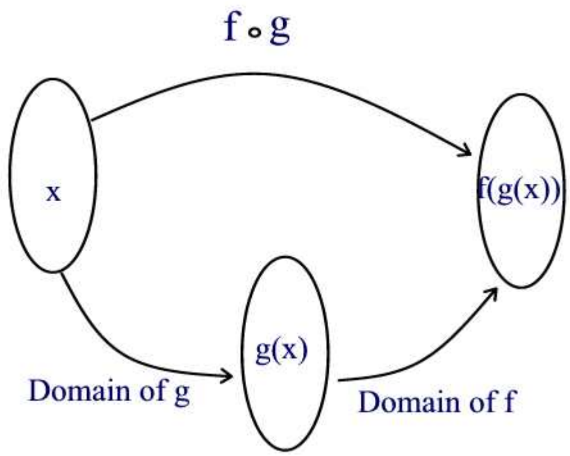


Figure 16

Composition of Functions Defined by Tables

Example 8.2
Complete the following table

x	0	1	2	3	4	5
f(x)	1	0	5	2	3	4
g(x)	5	2	3	1	4	8
f(g(x))						

Solution.

x	0	1	2	3	4	5
f(x)	1	0	5	2	3	4
g(x)	5	2	3	1	4	8
f(g(x))	4	5	2	0	3	undefined ■

Composition of Functions Defined by Formulas

Example 8.3
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $f(g(x))$ and $g(f(x))$.
(b) Calculate $f(g(5))$ and $g(f(-3))$.
(c) Are $f(g(x))$ and $g(f(x))$ equal?

Solution.
(a) $f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5$. Similarly, $g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$.
(b) $f(g(5)) = 2(5)^2 - 5 = 45$ and $g(f(-3)) = 4(-3)^2 + 4(-3) - 2 = 22$.
(c) $f(g(x)) \neq g(f(x))$.$\blacksquare$

With only one function you can build new functions using composition of the function with itself. Also, there is no limit on the number of functions that can be composed.

Example 8.4
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $f(f(x))$.
(b) Find $f(f(g(x)))$.

Solution.
(a) $f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$.
(b) $f(f(g(x))) = f(f(x^2 - 3)) = f(2x^2 - 5) = 2(2x^2 - 5) + 1 = 4x^2 - 9$.$\blacksquare$

Decomposition of Functions
If a formula for $f(g(x))$ is given then the process of finding the formulas for f and g is called **decomposition.**

Example 8.5
Decompose $f(g(x)) = \sqrt{1 - 4x^2}$.

Solution.
One possible answer is $f(x) = \sqrt{x}$ and $g(x) = 1 - 4x^2$. Another possible answer is $f(x) = \sqrt{1 - x^2}$ and $g(x) = 2x$.$\blacksquare$

Combinations of Functions
A second way to constructing new functions from old ones is to use the operations of addition, subtraction, multiplication, and division.
Let $f(x)$ and $g(x)$ be two given functions. Then for all x in the common

domain of these two functions we define new functions as follows:

- **Sum**: $(f+g)(x) = f(x) + g(x)$.
- **Difference**: $(f-g)(x) = f(x) - g(x)$.
- **Product**: $(f \cdot g)(x) = f(x) \cdot g(x)$.
- **Division**: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$.

In the following example we see how to construct the four functions discussed above when the individual functions are defined by formulas.

Example 8.6
Let $f(x) = x + 1$ and $g(x) = \sqrt{x+3}$. Find the common domain and then find a formula for each of the functions $f+g, f-g, f \cdot g, \frac{f}{g}$.

Solution.
The domain of $f(x)$ consists of all real numbers whereas the domain of $g(x)$ consists of all numbers $x \geq 3$. Thus, the common domain is the interval $[-3, \infty)$. For any x in this domain we have

$$\begin{array}{rcc}
(f+g)(x) & = & x+1+\sqrt{x+3} \\
(f-g)(x) & = & x+1-\sqrt{x+3} \\
(f \cdot g)(x) & = & x\sqrt{x+3}+\sqrt{x+3} \\
\left(\frac{f}{g}\right)(x) & = & \frac{x+1}{\sqrt{x+3}} \ \textit{provided}\ x > -3. \blacksquare
\end{array}$$

In the next example, we see how to evaluate the four functions when the individual functions are given in numerical forms.

Example 8.7
Suppose the functions f and g are given in numerical forms. Complete the following table:

x	-1	-1	0	1	1	3
f(x)	8	2	7	-1	-5	-3
g(x)	-1	-5	-11	7	8	9
$(f+g)(x)$						
$(f-g)(x)$						
$(f \cdot g)(x)$						
$(\frac{f}{g})(x)$						

Solution.

x	-1	-1	0	1	1	3
f(x)	8	2	7	-1	-5	-3
g(x)	-1	-5	-11	7	8	9
$(f+g)(x)$	7	-3	-4	6	3	6
$(f-g)(x)$	9	7	18	-8	-13	-12
$(f\cdot g)(x)$	-8	-10	-77	-7	-40	-27
$(\frac{f}{g})(x)$	-8	-$\frac{2}{5}$	-$\frac{7}{11}$	-$\frac{1}{7}$	-$\frac{5}{8}$	-$\frac{1}{3}$

Transformations of Functions
We close this section by giving a summary of the various transformations obtained when either the input or the output of a function is altered.

Vertical Shifts: The graph of $f(x)+k$ with $k>0$ is a vertical translation of the graph of $f(x)$, k units upward, whereas for $k<0$ it is a shift by k units downward.
Horizontal Shifts: The graph of $f(x+k)$ with $k>0$ is a horizontal translation of the graph of $f(x)$, k units to the left, whereas for $k<0$ it is a shift by k units to the right.
Reflections about the x-axis: For a given function $f(x)$, the graph of $-f(x)$ is a reflection of the graph of $f(x)$ about the x-axis.
Reflections about the y-axis: For a given function $f(x)$, the graph of $f(-x)$ is a reflection of the graph of $f(x)$ about the y-axis.
Vertical Stretches and Compressions: If a function $f(x)$ is given, then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of k for $k>1$, and a vertical compression for $0<k<1$.
What about $k<0$? If $|k|>1$ then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ followed by a reflection about the x-axis. If $0<|k|<1$ then the graph of $kf(x)$ is a vertical compression of the graph of $f(x)$ by a factor of k followed by a reflection about the x-axis.
Horizontal Stretches and Compressions: If a function $f(x)$ is given, then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ for $0<k<1$, and a horizontal compression for $k>1$.
What about $k<0$? If $|k|>1$ then the graph of $f(kx)$ is a horizontal compression of the graph of $f(x)$ followed by a reflection about the y-axis. If $0<|k|<1$ then the graph of $f(kx)$ is a horizontal stretch of the graph of

$f(x)$ by a factor of $\frac{1}{k}$ followed by a reflection about the y-axis.

Example 8.8
Write an equation for a graph obtained by vertically stretching the graph of $f(x) = x^2$ by a factor of 2, followed by a vertical upward shift of 1 unit.

Solution.
The function is given by the formula $y = 2f(x) + 1 = 2x^2 + 1.$ ■

Recommended Problems (pp. 54 - 56): 1, 5, 7, 13, 14, 15, 16, 17, 18, 19, 31.

9 Power and Polynomial Functions

A function $f(x)$ is a **power function** of x if there is a constant k such that

$$f(x) = kx^n$$

If $n > 0$, then we say that $f(x)$ is **proportional** to the nth power of x. If $n < 0$ then $f(x)$ is said to be **inversely proportional** to the nth power of x. We call k the **constant of proportionality**.

Example 9.1
(a) The strength, S, of a beam is proportional to the square of its thickness, h. Write a formula for S in terms of h.
(b) The gravitational force, F, between two bodies is inversely proportional to the square of the distance d between them. Write a formula for F in terms of d.

Solution.
(a) $S = kh^2$, where $k > 0$. (b) $F = \frac{k}{d^2}, k > 0.\blacksquare$

A power function $f(x) = kx^n$, with n a positive integer, is called a **monomial** function. A **polynomial** function is a sum of several monomial functions. Typically, a polynomial function is a function of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \quad a_n \neq 0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are all real numbers, called the **coefficients** of $f(x)$. The number n is a non-negative integer. It is called the **degree** of the polynomial. A polynomial of degree zero is just a constant function. A polynomial of degree one is a linear function, of degree two a quadratic function, etc. The number a_n is called the **leading coefficient** and a_0 is called the **constant term**.
Note that the terms in a polynomial are written in descending order of the exponents. Polynomials are defined for all values of x.

Example 9.2
Find the leading coefficient, the constant term and the degreee of the polynomial $f(x) = 4x^5 - x^3 + 3x^2 + x + 1$.

Solution.
The given polynomial is of degree 5, leading coefficient 4, and constant term 1.■

Remark 9.1
A polynomial function will never involve terms where the variable occurs in a denominator, underneath a radical, as an input of either an exponential or logarithmic function.

Example 9.3
Determine whether the function is a polynomial function or not:

(a) $f(x) = 3x^4 - 4x^2 + 5x - 10$
(b) $g(x) = x^3 - e^x + 3$
(c) $h(x) = x^2 - 3x + \frac{1}{x} + 4$
(d) $i(x) = x^2 - \sqrt{x} - 5$
(e) $j(x) = x^3 - 3x^2 + 2x - 5\ln x - 3$.

Solution.
(a) $f(x)$ is a polynomial function of degree 4.
(b) $g(x)$ is not a ploynomial degree because one of the terms is an exponential function.
(c) $h(x)$ is not a polynomial because x is in the denominator of a fraction.
(d) $i(x)$ is not a polynomial because it contains a radical sign.
(e) $j(x)$ is not a olynomial because one of the terms is a logarithm of x.■

Graphs of a Polynomial Function
Polynomials are continuous and smooth everywhere:

• A continuous function means that it can be drawn without picking up your pencil. There are no jumps or holes in the graph of a polynomial function.
• A smooth curve means that there are no sharp turns (like an absolute value) in the graph of the function.
• The y-intercept of the polynomial is the constant term a_0.

The shape of a polynomial depends on the degree and leading coefficient:

• If the leading coefficient, a_n, of a polynomial is positive, then the right hand side of the graph will rise towards $+\infty$.
• If the leading coefficient, a_n, of a polynomial is negative, then the right hand side of the graph will fall towards $-\infty$.
• If the degree, n, of a polynomial is even, the left hand side will do the same as the right hand side.
• If the degree, n, of a polynomial is odd, the left hand side will do the opposite of the right hand side.

Example 9.4
According to the graphs given below, indicate the sign of a_n and the parity of n for each curve.

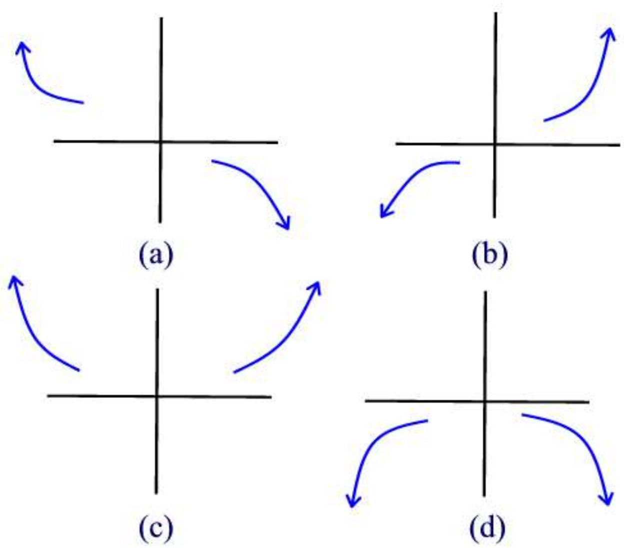


Figure 17

Solution.
(a) $a_n < 0$ and n is odd.
(b) $a_n > 0$ and n is odd.
(c) $a_n > 0$ and n is even.
(d) $a_n < 0$ and n is even.■

Long-Run Behavior of a Polynomial Function
If $f(x)$ and $g(x)$ are two functions such that $f(x) - g(x) \approx 0$ as x increases

without bound then we say that $f(x)$ resembles $g(x)$ in the **long run**. For example, if n is any positive integer then $\frac{1}{x^n} \approx 0$ in the long run.
Now, if $f(x) = a_n x^n + a_{n-1}x^n + \cdots + a_1 x + a_0$ then

$$f(x) = x^n(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}$$

Since $\frac{1}{x^k} \approx 0$ in the long run, for each $0 \leq k \leq n-1$ then

$$f(x) \approx a_n x^n$$

in the long run.

Example 9.5
Find the long run behavior of the polynomial $f(x) = 1 - 2x^4 + x^3$.

Solution.
The polynomial function $f(x) = 1 - 2x^4 + x^3$ resembles the function $g(x) = -2x^4$ in the long run.■

Zeros of a Polynomial Function
If f is a polynomial function in one variable, then the following statements are equivalent:

• $x = a$ is a **zero** or **root** of the function f.
• $x = a$ is a solution of the equation $f(x) = 0$.
• $(a, 0)$ is an x-intercept of the graph of f. That is, the point where the graph crosses the x-axis.

Example 9.6
Find the x-intercepts of the polynomial $f(x) = x^3 - x^2 - 6x$.

Solution.
Factoring the given function to obtain

$$\begin{aligned} f(x) &= x(x^2 - x - 6) \\ &= x(x-3)(x+2) \end{aligned}$$

Thus, the x-intercepts are the zeros of the equation

$$x(x-3)(x+2) = 0$$

That is, $x = 0, x = 3$, or $x = -2$.■

Recommended Problems (pp. 60 - 2): 1, 4, 7, 9, 13, 15, 17, 21, 25, 27, 30, 37.

10 Periodic Functions

A function f is said to be **periodic** if there is a smallest positive number p such that

$$f(x+p) = f(x)$$

for all x in the domain of f. For example, in trigonometry, one can show that the functions $f(x) = \sin x$ and $g(x) = \cos x$ are periodic of period 2π. Geometrically, this means that the graph is the same on any interval of length 2π. Figure 18 Shows the graph of $\sin x$ and Figure 19 shows the graph of $\cos x$.

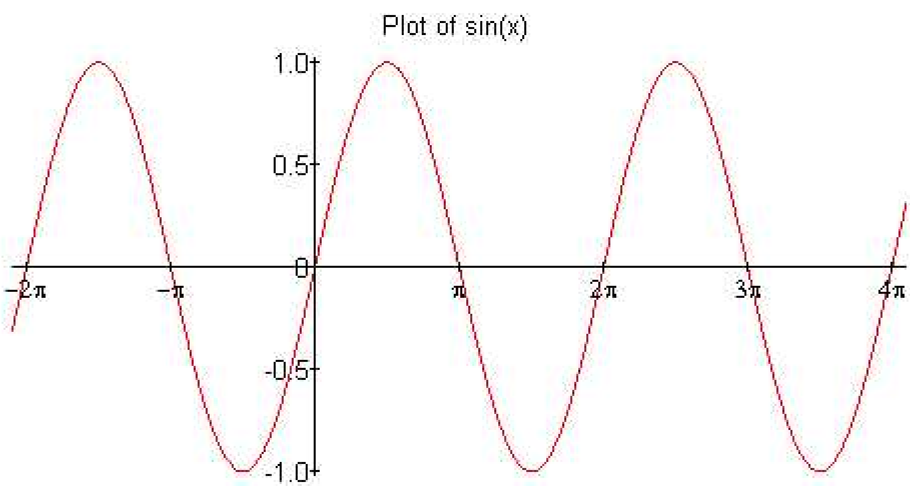


Figure 18

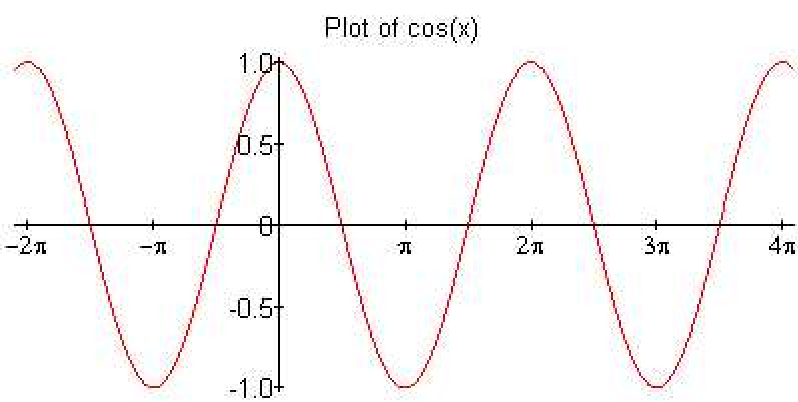


Figure 19

Amplitude and period of $y = a\sin(bx), y = a\cos(bx), b > 0$
We now consider graphs of functions that are transformations of the sine and cosine functions.

• **The parameter a:** This is outside the function and so deals with the output (i.e. the y values). Since $-1 \leq \sin(bx) \leq 1$ and $-1 \leq \cos(bx) \leq 1$ then $-a \leq a\sin(bx) \leq a$ and $-a \leq a\cos(bx) \leq a$. So, the range of the function $y = a\sin(bx)$ or the function $y = a\cos(bx)$ is the closed interval $[-a, a]$. The number $|a|$ is called the **amplitude**. Graphically, this number describes how tall the graph is. The amplitude is half the distance from the top of the curve to the bottom of the curve. If $b = 1$, the amplitude $|a|$ indicates a vertical stretch of the basic sine or cosine curve if $a > 1$, and a vertical compression if $0 < a < 1$. If $a < 0$ then a reflection about the x-axis is required.
Figure 20 shows the graph of $y = 2\sin x$ and the graph of $y = 3\sin x$.

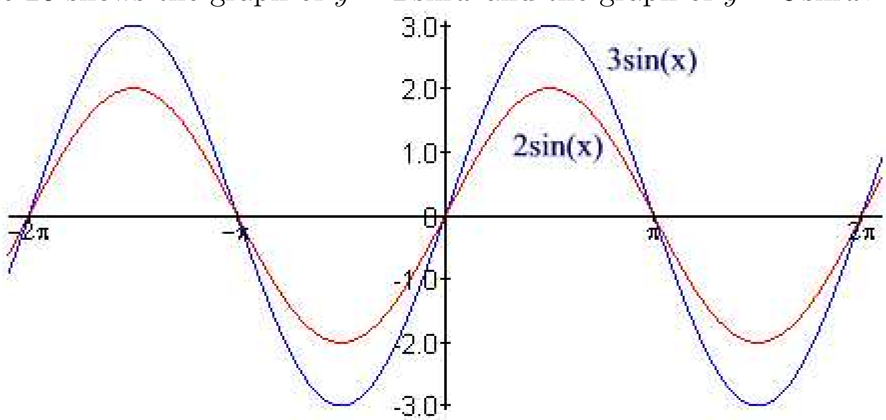


Figure 20

• **The parameter b:** This is inside the function and so effects the input (i.e. x values). Now, the graph of either $y = a\sin(bx)$ or $y = a\cos(bx)$ completes one period from $bx = 0$ to $bx = 2\pi$. By solving for x we find the interval of one period to be $[0, \frac{2\pi}{b}]$. Thus, the above mentioned functions have a period of $\frac{2\pi}{b}$. The number b tells you the number of cylces in the interval $[0, 2\pi]$. Graphically,b either stretches (if $b < 1$) or compresses (if $b > 1$) the graph horizontally.
Figure 21 shows the function $y = \sin x$ with period 2π and the function

$y = \sin(2x)$ with period π.

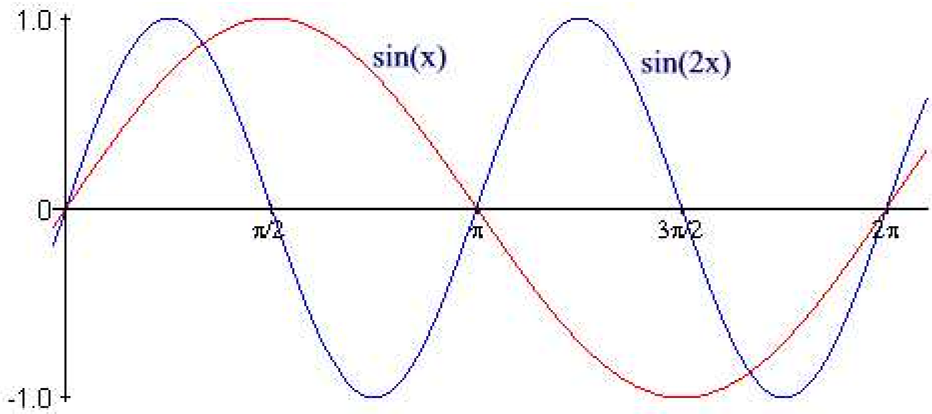


Figure 21

Recommended Problems (pp. 67 - 9): 3, 5, 7, 9, 10, 11, 13, 14, 17, 19, 20, 24, 25.

11 Instantaneous Rate of Change

In this section, we discuss the concept of the instantaneous rate of change of a given function. As an application, we use the velocity of a moving object. The motion of an object along a line at a particular instant is very difficult to define precisely. The modern approach consists of computing the average velocity over smaller and smaller time intervals. To be more precise, let $s(t)$ be the **position** function or displacement of a moving object at time t. We would like to compute the velocity of the object at the instant $t = t_0$:

Average Velocity
We start by finding the **average velocity** of the object over the time interval $t_0 \leq t \leq t_0 + \Delta t$ given by the expression

$$\overline{v} = \frac{Distance\ Traveled}{Elapsed\ Time} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}$$

Geometrically, the average velocity over the time interval $[t_0, t_0 + \Delta t]$ is just the slope of the line joining the points $(t_0, s(t_0))$ and $(t_0 + \Delta t, s(t_0 + \Delta t))$ on the graph of $s(t)$.(See Figure 22)

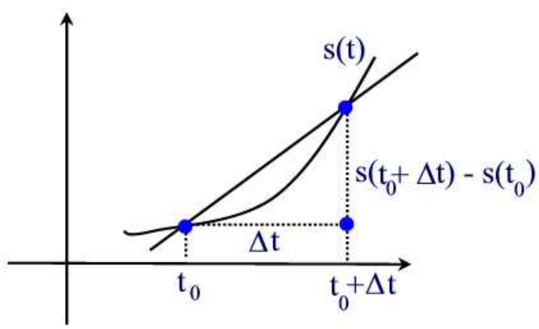


Figure 22

Example 11.1
A freely falling body experiencing no air resistance falls $s(t) = 16t^2$ feet in t seconds. Complete the following table

time interval	[1.8,2]	[1.9,2]	[1.99,2]	[1.999,2]	[2,2.0001]	[2,2.001]	[2,2.01]
Average velocity							

Solution.

time interval	[1.8,2]	[1.9,2]	[1.99,2]	[1.999,2]	[2,2.0001]	[2,2.001]	[2,2.01]
Average velocity	60.8	62.4	63.84	63.98	64.0016	64.016	64.16

Instantaneous Velocity and Speed
The next step is to calculate the average velocity on smaller and smaller time intervals (that is, make Δt close to zero). The average velocity in this case approaches what we would intuitively call the **instantaneous velocity** at time $t = t_0$ which is defined using the **limit** notation by

$$v(t_0) = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}$$

Geometrically, the instantaneous velocity at t_0 is the slope of the tangent line to the graph of $s(t)$ at the point $(t_0, s(t_0))$.(See Figure 23)

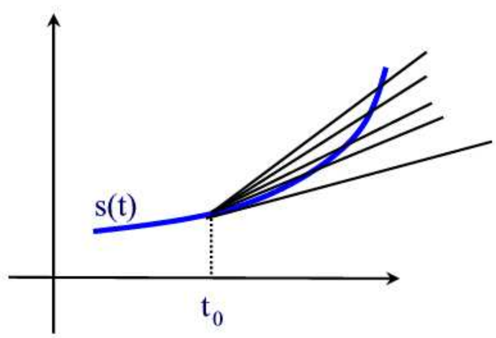


Figure 23

Example 11.2
For the distance function in Example 11.1, find the instantaneous velocity at $t = 2$.

Solution.
Examining the bottom row of the table in Example 11.1, we see that the average velocity seems to be approaching the value 64 as we shrink the time intervals. Thus, it is reasonable to expect the velocity to be $v(2) = 64\ ft/sec.$■

In general, we define the **instantaneous rate of change** of a function $y = f(x)$ at $x = a$ to be

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We define the **speed** of a moving object to be the absolute value of the velocity function. Sometimes there is confusion between the words "speed" and "velocity". Speed is a nonnegative number that indicates how fast an object is moving, whereas velocity indicates both speed and direction(relative to a coordinate system). For example, if the object is moving along a vertical line we define a positive velocity when the object is going upward and a negative velocity when the object is going downward.

The instantaneous rate of change of a function $f(x)$ from $x = a$ to $x = a + h$ is called the **derivative of f at a** and we denote it by $f'(a)$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the derivative exists, i.e. can be found, then we say that the function is **differentiable** at $x = a$. The process of finding the derivative of a function is called **differentiation**. If a function has no derivative at a point then we say that it is non-differentiable there.
Since the instantaneous rate of change represents the slope of a tangent line then $f'(a)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$. The equation of the tangent line is given by

$$y - f(a) = f'(a)(x - a).$$

Example 11.3
(a) Find $f'(1)$ for $f(x) = x^2$.
(b) Find the equation of the tangent line to the graph of $f(x)$ at the point $(1, f(1))$.

Solution.
Completing the following chart

x	[0.9,1]	[0.99,1]	[0.999,1]	[1,1.0001]	[1,1.001]	[1,1.01]	[1,1.1]
$\frac{f(b)-f(a)}{b-a}$	1.9	1.99	1.999	2.0001	2.001	2.01	2.1

we see that $f'(1) = 2.$
(b) The equation of the tangent line is

$$y - f(1) = f'(1)(x - 1)$$

or

$$y - 1 = 2(x - 1)$$

In point-intercept form, we have $y = 2x - 1.$ ■

Example 11.4 *(Numerical Estimation of the Derivative)*
Find approximate values for $f'(x)$ at each of the x-values given in the following table

x	0	5	10	15	20
f(x)	100	70	55	46	40

Solution.
The derivative can be estimated by using the average rate of change or the difference quotient

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

If a is a left-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(b) - f(a)}{b - a}$$

where $b > a$. If a is a right-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(a) - f(b)}{a - b}$$

where $b < a$. If a is an interior point then $f'(a)$ is estimated by

$$f'(a) \approx \frac{1}{2}\left(\frac{f(a) - f(b)}{a - b} + \frac{f(c) - f(a)}{c - a}\right)$$

where $b < a < c$. For example,

$$\begin{array}{lllll}
f'(0) & \approx & \frac{f(5)-f(0)}{5} & = & -6 \\
f'(5) & \approx & \frac{1}{2}\left(\frac{f(10)-f(5)}{5} + \frac{f(5)-f(0)}{5}\right) & = & -4.5 \\
f'(10) & \approx & \frac{1}{2}\left(\frac{f(15)-f(10)}{5} + \frac{f(10)-f(5)}{5}\right) & = & -2.4 \\
f'(15) & \approx & \frac{1}{2}\left(\frac{f(20)-f(15)}{5} + \frac{f(15)-f(10)}{5}\right) & = & -1.5 \\
f'(20) & \approx & \frac{f(20)-f(15)}{5} & = & -1.2
\end{array}$$

The quantity $\frac{f(10)-f(5)}{10-5}$ is known as the right slope estimation of $f'(5)$. Similarly, we can estimate $f'(5)$ by using a left slope estimation,i.e.

$$f'(5) \approx \frac{f(5)-f(0)}{5-0} = -6$$

An improved estimation consists of taking the average of the left slope and the right slope, that is,

$$f'(5) \approx \frac{-3-6}{2} = -4.5 \blacksquare$$

Recommended Problems (pp. 99 - 101): 1, 3, 4, 6, 8, 10, 11, 12, 13, 15, 18, 19, 24.

12 The Derivative Function

Recall that a function f is differentiable at x if the following limit exists

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

Thus, we associate with the function f, a new function f' whose domain is the set of points x at which the limit (2) exists. We call the function f' the **derivative function** of f.

The Derivative Function Graphically
Since the derivative at a point represents the slope of the tangent line then one can obtain the graph of the derivative function from the graph of the original function. It is important to keep in mind the relationship between the graphs of f and f'. If $f'(x) > 0$ then the tangent line must be tilted upward and the graph of f is rising or increasing. Similarly, if $f'(x) < 0$ then the tangent line is tilted downward and the graph of f is falling or decreasing. If $f'(a) = 0$ then the tangent line is horizontal at $x = a$.

Example 12.1
Sketch the graph of the derivative of the function shown in Figure 24.

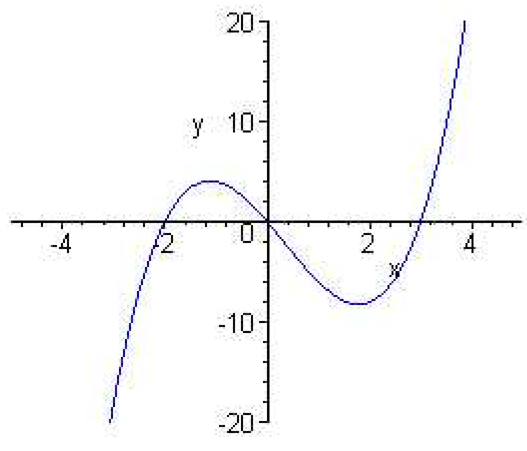


Figure 24

Solution.
Note that for $x < -1.12$ the derivative is positive and getting less and less positive. At $x \approx -1.12$ we have $f'(-1.12) = 0$. For $-1.12 < x < 0$ the

derivative is negative and getting more and more negative till reaching $x = 0$. For $0 < x < 1.79$ the derivative is less and less negative and at $x = 1.79$ we have $f'(1.79) = 0$. Finally, for $x > 1.79$ the derivative is getting more and more positive. Thus, a possible graph of f' is given in Figure 25. ■

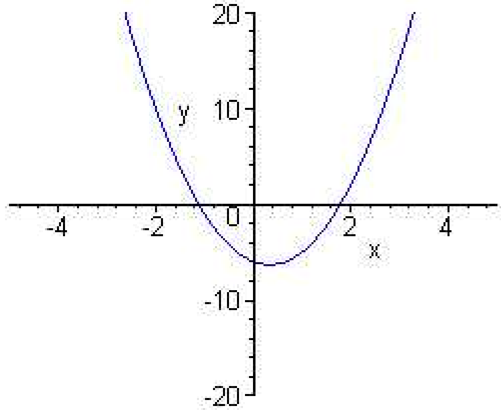


Figure 25

The Derivative Function Numerically

Here, we want to estimate the derivative of a function defined by a table. The derivative can be estimated by using the average rate of change or the difference quotient

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

If a is a left-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(b) - f(a)}{b - a}$$

where $b > a$. If a is a right-endpoint then $f'(a)$ is estimated by

$$f'(a) \approx \frac{f(a) - f(b)}{a - b}$$

where $b < a$. If a is an interior point then $f'(a)$ is estimated by

$$f'(a) \approx \frac{1}{2}\left(\frac{f(a) - f(b)}{a - b} + \frac{f(c) - f(a)}{c - a}\right)$$

where $b < a < c$.

Example 12.2
Find approximate values for $f'(x)$ at each of the x-values given in the following table

x	0	5	10	15	20
f(x)	100	70	55	46	40

Solution.

$$\begin{array}{rcrcr} f'(0) & \approx & \frac{f(5)-f(0)}{5} & = & -6 \\ f'(5) & \approx & \frac{1}{2}\left(\frac{f(10)-f(5)}{5}+\frac{f(5)-f(0)}{5}\right) & = & -4.5 \\ f'(10) & \approx & \frac{1}{2}\left(\frac{f(15)-f(10)}{5}+\frac{f(10)-f(5)}{5}\right) & = & -2.4 \\ f'(15) & \approx & \frac{1}{2}\left(\frac{f(20)-f(15)}{5}+\frac{f(15)-f(10)}{5}\right) & = & -1.5 \\ f'(20) & \approx & \frac{f(20)-f(15)}{5} & = & -1.2 \blacksquare \end{array}$$

The Derivative Function From a Formula
Now, if a formula for f is given then by applying the definition of $f'(x)$ as the limit of the difference quotient we can find a formula of f' as shown in the following two problems.

Example 12.3 *(Derivative of a Constant Function)*
Suppose that $f(x) = k$ for all x. Find a formula for $f'(x)$.

Solution.

$$\begin{array}{rcl} f'(x) & = & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ & = & \lim_{h\to 0} \frac{k-k}{h} = 0. \end{array}$$

Thus, $f'(x) = 0.\blacksquare$

Example 12.4 *(Derivative of a Linear Function)*
Find the derivative of the linear function $f(x) = mx + b$.

Solution.

$$\begin{array}{rcl} f'(x) & = & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ & = & \lim_{h\to 0} \frac{m(x+h)+b-(mx+b)}{h} \\ & = & \lim_{h\to 0} \frac{mh}{h} = m. \end{array}$$

Thus, $f'(x) = m.\blacksquare$

Recommended Problems (pp. 104 - 6): 5, 6, 7, 9, 11, 13, 15, 19, 21.

13 Leibniz Notation for The Derivative

When dealing with mathematical models that involve derivatives it is convenient to denote the prime notation of the derivative of a function $y = f(x)$ by $\frac{dy}{dx}$. That is,

$$\frac{dy}{dx} = f'(x)$$

This notation is called **Leibniz notation** (due to W.G. Leibniz). For example, we can write $\frac{dy}{dx} = 2x$ for $y' = 2x$.
When using Leibniz notation to denote the value of the derivative at a point a we will write

$$\left.\frac{dy}{dx}\right|_{x=a}$$

Thus, to evaluate $\frac{dy}{dx} = 2x$ at $x = 2$ we would write

$$\left.\frac{dy}{dx}\right|_{x=2} = 2x|_{x=2} = 2(2) = 4.$$

Remark 13.1
Even though $\frac{dy}{dx}$ appears as a fraction but it is not. It is just an alternative notation for the derivative. A concept called **differential** will provide meaning to symbols like dy and dx.

One of the advantages of Leibniz notation is the recognition of the units of the derivative. For example, if the position function $s(t)$ is expressed in meters and the time t in seconds then the units of the velocity function $\frac{ds}{dt}$ are meters/sec.
In general, the units of the derivative are the units of the dependent variable divided by the units of the independent variable.

Example 13.1
The cost, C (in dollars) to produce x gallons of ice cream can be expressed as $C = f(x)$. What are the units of measurements and the meaning of the statement $\left.\frac{dC}{dx}\right|_{x=200} = 1.4$?

Solution.
$\frac{dC}{dx}$ is measured in dollars per gallon. The notation

$$\left.\frac{dC}{dx}\right|_{x=200} = 1.4$$

means that if 200 gallons of ice cream have already been produced then the cost of producing the next gallon will be roughly 1.4 dollars.■

Example 13.2
The derivative of the velocity function v is called **acceleration** and is denoted by a. Suppose that v is measured in *meters/seconds*, what are the units of a?

Solution.
The units of a are $meters/seconds/seconds = meters/seconds^2$.■

Local Linear Approximation
Finally, one can use the derivative at a point to approximate values of the function at nearby points. For example, if we know the values of $f(a)$ and $f'(a)$ then for a nearby point b the value of $f(b)$ is found by the formula

$$f(b) \approx f'(a)(b-a) + f(a).$$

Example 13.3
Climbing health care costs have been a source of concern for some time. Use the data in the table below to estimate the average per capita expenditure in 1991 and 2010 assuming that the costs climb at the same rate since 1990.

Year	1970	1975	1980	1985	1990
Per capita expenditure ($)	349	591	1055	1596	2714

Solution.
Between 1985 and 1990 the rate of increase in the costs is $\frac{2714-1596}{5} = \$223.60$ per year. Since we are assuming that the costs continue to increase at the same rate then

$$C(1991) \approx C(1990) + C'(1990)(1991-1990) = 2714 + 223.60 = \$2937.60$$

and

$$C(2010) = 2714 + 223.60(10) = \$7,186.$$■

Recommended Problems (pp. 111 - 3): 1, 3, 6, 8, 9, 12, 13, 14, 17, 23.

14 The Second Derivative

Let $f(x)$ be a differentiable function. If the limit

$$\lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

exists then we say that the function $f'(x)$ is differentiable and we denote its derivative by $f''(x)$ or using Leibniz notation

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

We call $f''(x)$ the **second derivative** of $f(x)$.

Now, recall that if $f'(x) > 0$ (resp. $f'(x) < 0$) over an interval I then the function $f(x)$ is increasing (resp. decreasing on I). So if $f''(x) > 0$ on I then $f'(x)$ is increasing on I. So either $f'(x)$ gets more and more positive or less and less negative. This occurs only when the graph of f is concave up. Similarly, if $f''(x) < 0$ on I then $f'(x)$ is decreasing. So either $f(x)$ is getting less and less positive or more and more negative. This means that the graph of f is concave down.

Remark 14.1
Note that when a curve is concave up then the tangent lines lie below the curve whereas when it is concave down then the tangent lines lie above the curve.

Example 14.1
Give the signs of f' and f'' for the function whose graph is given in Figure 26

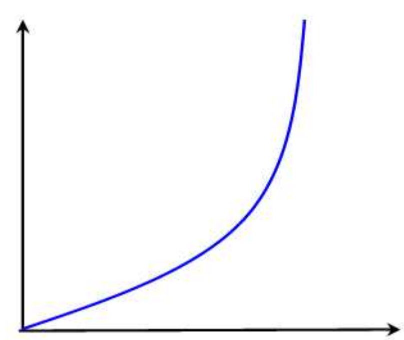


Figure 26

Solution.
Since f is always increasing then f' is always positive. Since the graph is concave up then f'' is always positive.$\blacksquare$

Example 14.2
Find where the graph of $f(x) = x^3 + 3x + 1$ is concave up and where it is concave down.

Solution.
Finding the first and second derivatives of f we obtain $f'(x) = 3x^2 + 3$ and $f''(x) = 6x$. Thus, the graph of f is concave up for $x > 0$ and concave down for $x < 0$.$\blacksquare$

As an application to the second derivative, we consider the motion of an object determined by the position function $s(t)$. Recall that the velocity of the object is defined to be the first derivative of $s(t)$, i.e.

$$v(t) = s'(t) = \frac{ds}{dt}$$

and the absolute value of $v(t)$ is the speed. When the object speeds up we say that he/she accelerates and when the object slows down we say that he/she decelerates. We define the **acceleration** of an object as the derivative of the velocity function and consequently as the second derivative of the position function

$$a(t) = \frac{d^2s}{dt^2} = \frac{ds}{dt}.$$

Example 14.3
A particle is moving along a straight line. If its distance, s, to the right of a fixed point is given by Figure 27, estimate:

(a) When the particle is moving to the right and when it is moving to the left.
(b) When the particle has positive acceleration and when it has negative

acceleration.

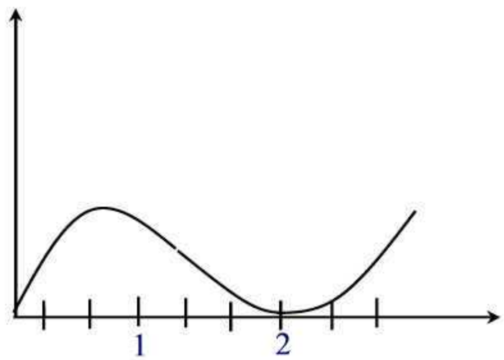


Figure 27

Solution.
(a) When s is increasing then the particle moves to the right. This occurs when $0 < t < \frac{2}{3}$ and for $t > 2$. On the other hand, the particle moves to the left when s is decreasing. This happens when $\frac{2}{3} < t < 2$.
(b) Positive acceleration occurs when the graph is concave up. This occurs when $t > \frac{4}{3}$. The particle has negative acceleration when the curve is concave down, i.e for $t < \frac{4}{3}$.■

Recommended Problems (pp. 115 - 7): 2, 3, 8, 9, 11, 12, 13, 15, 17, 18, 23.

15 Marginal Cost and Revenue

We start this section by looking at possible graphs of the cost and revenue functions.
A cost function can be linear as shown in Figure 28(a) , or have the shape shown in Figure 28(b). Note that in Figure 28(b), the graph is concave up then concave down. This means that the cost function increases first at a slow rate, then slows down, and finally increases at a faster rate.

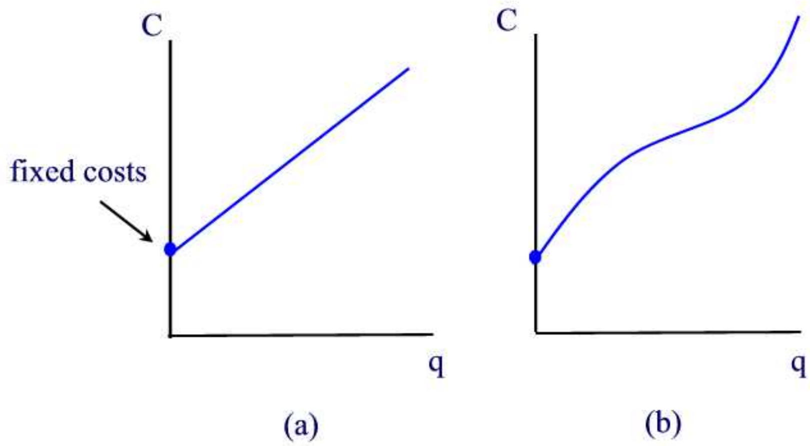


Figure 28

Now, since $R = pq$ then the graph of R as a function of q is a straight line going through the origin and with slope p when the price p is constant (See Figure 29(a)), or the graph shown in Figure 29(b).

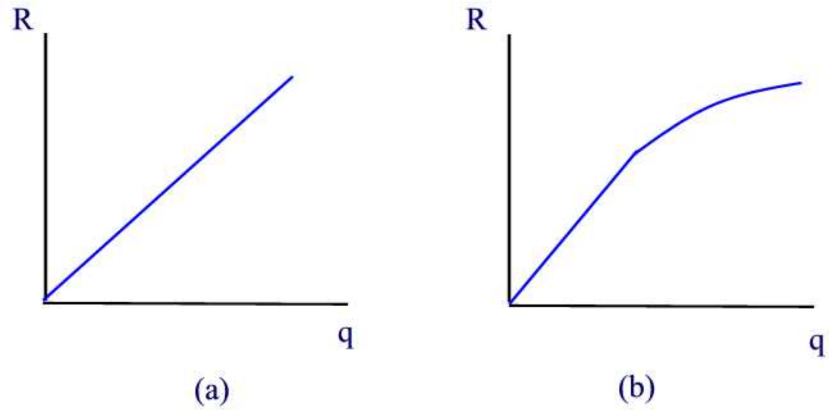


Figure 29

Marginal Analysis

Marginal analysis is an area of economics concerned with estimating the effect on quantities such as cost, revenue, and profit when the level of production is changed by a unit amount. For example, if $C(q)$ is the cost of producing q units of a certain commodity, then the **marginal cost**, MC(q), is the additional cost of producing one more unit and is given by the difference $MC(q) = C(q+1) - C(q)$. Using the estimation

$$C'(q) \approx \frac{C(q+1) - C(q)}{(q+1) - q} = C(q+1) - C(q)$$

we find that

$$MC(q) \approx C'(q)$$

and for this reason, we will compute the marginal cost by the derivative $C'(q)$.

Similarly, if $R(q)$ is the revenue obtained from producing q units of a commodity, then the **marginal revenue**, MR(q), is the additional revenue obtained from producing one more unit, and we compute $MR(q)$ by the derivative $R'(q)$.

Example 15.1

Let $C(q)$ represent the cost, $R(q)$ the revenue, and $P(q)$ the total profit, in dollars, of producing q units.

(a) If $C'(50) = 75$ and $R'(50) = 84$, approximately how much profit is earned by the 51st item?
(b) If $C'(90) = 71$ and $R'(90) = 68$, approximately how much profit is earned by the 91st item?

Solution.
(a) $P'(50) = R'(50) - C'(50) = 84 - 75 = 9$.
(b) $P'(90) = R'(90) - C'(90) = 68 - 71 = -3$. A loss by 3 dollars.■

Example 15.2
Cost and Revenue are given in Figure 30. Sketch the graphs of the marginal cost and marginal revenue.

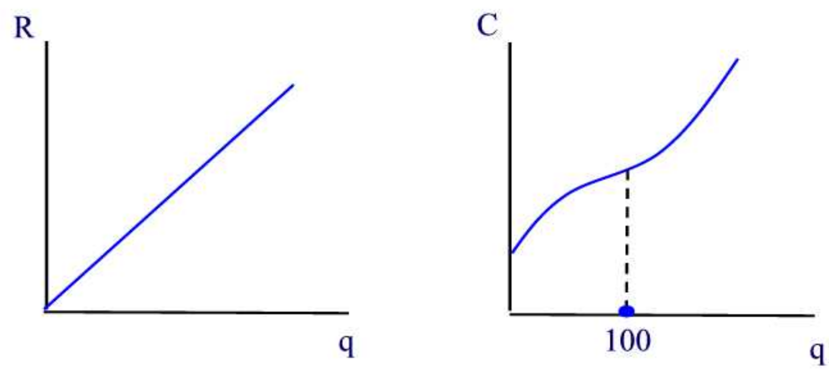


Figure 30

Solution.
Since the graph of R is a straight line with positive slope p then the graph of MR is a horizontal line at p. (See Figure 31 (a)). For the marginal cost, note that the marginal cost is decreasing for $q < 100$ and then increasing for $q > 100$. Thus, $q = 100$ is a minimum point. (See Figure 31 (b)).■

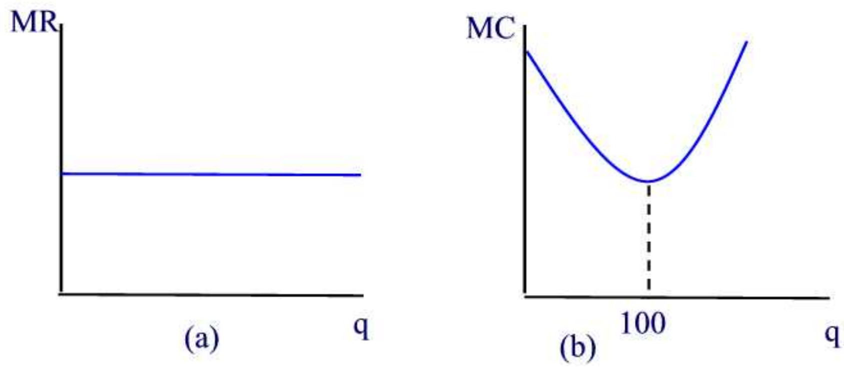


Figure 31

Maximizing Profit
We end this section by considering the question of maximizing the profit function P. That is, maximizing the function

$$P(q) = R(q) - C(q).$$

We will see in Section 24, that the profit function attains its maximum for the level of production q for which $P'(q) = 0, i.e. MC(q) = MR(q)$. Geometrically, this occurs at q where the tangent line to the graph of C is parallel to the tangent line to the graph of R at q.

Example 15.3
A manufacturer estimates that when q units of a particular commodity are produced each month, the total cost (in dollars) will be

$$C(q) = \frac{1}{8}q^2 + 4q + 200$$

and all units are sold at a price $p = 49 - q$ dollars per unit. Determine the price that corresponds to the maximum profit.

Solution.
The revenue function is given by

$$R(q) = pq = 49q - q^2$$

and its derivative is $MR(q) = 49 - 2q$. Setting this expression equal to the marginal cost to obtain

$$\frac{1}{4}q + 4 = 49 - 2q.$$

Solving for q we obtain $q = 20$ units. Thus, $p = 49 - 20 = \$29$.■

Example 15.4
Locate the quantity in Figure 32 where the profit function is maximum.

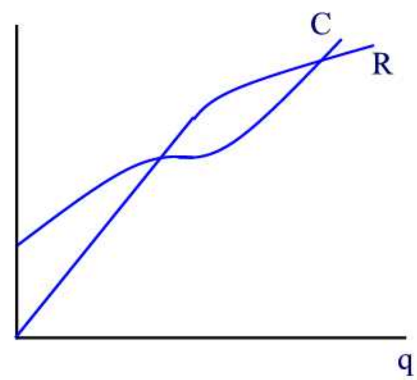


Figure 32

Solution.
The quantity q' for which profit is maximized is shown in Figure 33.■

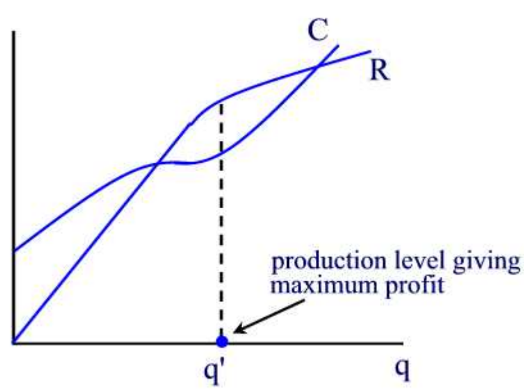


Figure 33

Recommended Problems (pp. 122 - 4): 1, 3, 5, 7, 8, 9, 13.

16 Derivative Formulas for Power and Polynomials

Finding the derivative function by using the limit of the difference quotient is sometimes difficult for functions with complicated expressions. Fortunately, there is an indirect way for computing derivatives that does not compute limits but instead uses formulas which we will derive in this section and in the coming sections.
We first derive a couple of formulas of differentiation.

Theorem 16.1
If f is differentiable and k is a constant then the new function $kf(x)$ is differentiable with derivative given by

$$[kf(x)]' = kf'(x).$$

Proof.

$$\begin{array}{rcl} [kf'(x) & = & \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h} = \lim_{h\to 0}\frac{k(f(x+h)-f(x))}{h} \\ & = & k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = kf'(x) \end{array}$$

where we used the fact that a constant can be taking across the limit sign by the properties of limits.■

Theorem 16.2
If $f(x)$ and $g(x)$ are two differentiable functions then the functions $f+g$ and $f-g$ are also differentiable with derivatives

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

Proof.
Again by using the definition of the derivative and the fact that the limit of a sum/difference is the sum/difference of limits we find

$$\begin{array}{rcl} [f(x)+g(x)]' & = & \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} \\ & = & \lim_{h\to 0}\frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h} \\ & = & \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x). \end{array}$$

The same proof is valid for the difference formula.■

Next, we state and give a partial proof of a rule for finding the derivative of a power function of the form $f(x) = x^n$.

Theorem 16.3 *(Power Rule)*
For any real number n, the derivative of the function $y = x^n$ is given by the formula

$$\frac{dy}{dx} = nx^{n-1}$$

Proof.
We prove the result when n is a positive integer. We start by writing the definition of the derivative of any function $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Letting $h = ax - x$ we can rewrite the previous definition in the form

$$f'(x) = \lim_{a \to 1} \frac{f(ax) - f(x)}{ax - x}.$$

Thus,

$$f'(x) = \lim_{a \to 1} \frac{(ax)^n - x^n}{ax - x} = x^{n-1} \lim_{a \to 1} \frac{a^n - 1}{a - 1}.$$

Dividing $a^n - 1$ by $a - 1$ by the method of synthetic division we find

$$a^n - 1 = (a-1)(1 + a + a^2 + a^3 + \cdots + a^{n-1}).$$

Thus,

$$f'(x) = x^{n-1} \lim_{a \to 1}(1 + a + a^2 + \cdots + a^{n-1}) = nx^{n-1}.\blacksquare$$

Example 16.1
Use the power rule to differentiate the following:

(a) $y = x^{\frac{4}{3}}$ (b) $y = \frac{1}{\sqrt[3]{x}}$ (c) $y = x^{\pi}$.

Solution.
(a) Using the power rule with $n = \frac{4}{3}$ to obtain $y' = \frac{4}{3}x^{\frac{1}{3}}$.
(b) Since $y = x^{-\frac{1}{3}}$ then using the power rule with $n = -\frac{1}{3}$ to obtain $y' = -\frac{1}{3}x^{-\frac{4}{3}}$.
(c) Using the power rule with $n = \pi$ to obtain $y' = \pi x^{\pi - 1}$.$\blacksquare$

Remark 16.1
The derivative of a function of the form $y = 2^x$ is not $y' = x2^{x-1}$ because $y = 2^x$ is an exponential function and not a power function. A formula for finding the derivative of an exponential function will be discussed in the next section.

Now, combining the results discussed above, we can find the derivative of functions that are combinations of power functions of the form ax^n. In particular, the derivative of a polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is given by the formula

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

Example 16.2
Find the derivative of the function $y = \sqrt{3}x^7 - \frac{x^5}{5} + \pi$.

Solution.
The derivative is $f'(x) = 7\sqrt{3}x^6 - x^4$.■

Example 16.3
Find the second derivative of $y = 5\sqrt[3]{x} - \frac{10}{x^4} + \frac{1}{2\sqrt{x}}$.

Solution.
Note that the given function can be written in the form $y = 5x^{\frac{1}{3}} - 10x^{-4} + \frac{1}{2}x^{-\frac{1}{2}}$. Thus, the first derivative is

$$y' = \frac{5}{3}x^{-\frac{2}{3}} + 40x^{-5} - \frac{1}{4}x^{-\frac{3}{2}}.$$

The second derivative is

$$y'' = -\frac{10}{9}x^{-\frac{5}{3}} - 200x^{-6} + \frac{3}{8}x^{-\frac{5}{2}}.■$$

Recommended Problems (pp. 141 - 2): 1, 3, 6, 11, 18, 29, 32, 34, 36, 38, 41, 42.

17 Derivative Formulas for Exponential and Logarithmic Functions

We start this section by looking at the limit

$$\lim_{h\to 0}\frac{e^h-1}{h}.$$

The chart below suggests that the limit is 1.

h	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$\frac{e^h-1}{h}$	0.995	0.9995	0.99995	undefined	1.0000	1.0005	1.005

Now, let's try and find the derivative of the function $f(x) = e^x$ at any number x. By the definition of the derivative and the limit above we see that

$$\begin{array}{rcl}
f'(x) & = & \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
& = & \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\
& = & \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\
& = & e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x.
\end{array}$$

This means that e^x is its own derivative:

$$\frac{d}{dx}(e^x) = e^x.$$

Now, suppose that the x in e^x is replaced by a differentiable function of x, say $u(x)$. We would like to find the derivative of e^u with respect to x, i.e., what is $\frac{d}{dx}(e^u)$?

Theorem 17.1

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}.$$

Proof.
By the definition of the derivative we have

$$\frac{d}{dx}(e^u) = \lim_{h\to 0}\frac{e^{u(x+h)}-e^{u(x)}}{h}.$$

Since u is differentiable at x then by letting

$$v = \frac{u(x+h) - u(x)}{h} - u'(x)$$

we find

$$u(x+h) = u(x) + (v + u'(x))h$$

with $\lim_{h\to 0} v = 0$. Similarly, we can write

$$e^{y+k} = e^y + (w + e^y)k$$

with $\lim_{k\to 0} w = 0$. In particular, letting $y = u(x)$ and $k = (v + u'(x))h$ we find

$$e^{u(x)+(v+u'(x))h} = e^{u(x)} + (w + e^{u(x)})(v + u'(x))h.$$

Hence,

$$\begin{array}{rcl} e^{u(x+h)} - e^{u(x)} & = & e^{u(x)+(v+u'(x))h} - e^{u(x)} \\ & = & e^{u(x)} + (w + e^{u(x)})(v + u'(x))h - e^{u(x)} \\ & = & (w + e^u)(v + u'(x))h \end{array}$$

Thus,

$$\begin{array}{rcl} \frac{d}{dx}(e^u) & = & \lim_{h\to 0} \frac{e^{u(x+h)} - e^{u(x)}}{h} \\ & = & \lim_{h\to 0}(w + e^u)(v + u'(x)) \\ & = & e^u u' \ \blacksquare \end{array}$$

Next, we want to find the derivative of the function $f(x) = a^x$, where $a > 0$ and $a \neq 1$. First, note that $f(x) = a^x = (e^{\ln a})^x = e^{x \ln a}$. Thus, by Theorem 16.1 we see that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x\ln a}\frac{d}{dx}(x\ln a) = a^x \ln a.$$

Example 17.1
Find the derivative of each of the following functions:

(a) $f(x) = 3^x$ (b) $y = 2 \cdot 3^x + 5 \cdot e^{3x-4}$.

Solution.

(a) $f'(x) = 3^x \ln 3$.

(b) $y' = 2(3^x)' + 5(e^{3x-4})' = 2 \cdot 3^x \ln 3 + 5(3)e^{3x-4} = 2 \cdot 3^x \ln 3 + 15 \cdot e^{3x-4}$.■

We end this section, by finding the derivative of the function $f(x) = \ln x$. In the next section, we will prove the formula

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Recommended Problems (pp. 145 - 6): 1, 5, 11, 15, 16, 21, 25, 28, 31, 32.

18 Derivatives of Composite Functions: The Chain Rule

In this section we want to find the derivative of a composite function $f(g(x))$ where $f(x)$ and $g(x)$ are two differentiable functions.

Theorem 18.1
If f and g are differentiable then $f(g(x))$ is differentiable with derivative given by the formula

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

This result is known as the **chain rule**. Thus, the derivative of $f(g(x))$ is the derivative of $f(x)$ evaluated at $g(x)$ times the derivative of $g(x)$.

Proof.
By the definition of the derivative we have

$$\frac{d}{dx}f(g(x)) = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}.$$

Since g is differentiable at x then by letting

$$v = \frac{g(x+h) - g(x)}{h} - g'(x)$$

we find

$$g(x+h) = g(x) + (v + g'(x))h$$

with $\lim_{h\to 0} v = 0$. Similarly, we can write

$$f(y+k) = f(y) + (w + f'(y))k$$

with $\lim_{k\to 0} w = 0$. In particular, letting $y = g(x)$ and $k = (v + g'(x))h$ we find

$$f(g(x) + (v + g'(x))h) = f(g(x)) + (w + f'(g(x)))(v + g'(x))h.$$

Hence,

$$\begin{array}{rcc} f(g(x+h)) - f(g(x)) &=& f(g(x) + (v + g'(x))h) - f(g(x)) \\ &=& f(g(x)) + (w + f'(g(x)))(v + g'(x))h - f(g(x)) \\ &=& (w + f'(g(x)))(v + g'(x))h \end{array}$$

Thus,

$$\begin{array}{rcl} \frac{d}{dx}f(g(x)) & = & \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} \\ & = & \lim_{h\to 0}(w+f'(g(x)))(v+g'(x)) \\ & = & f'(g(x))g'(x). \end{array}$$

This completes a proof of the theorem. ■

Example 18.1
Find the derivative of $y = (4x^2 + 1)^7$.

Solution.
First note that $y = f(g(x))$ where $f(x) = x^7$ and $g(x) = 4x^2 + 1$. Thus, $f'(x) = 7x^6$, $f'(g(x)) = 7(4x^2+1)^6$ and $g'(x) = 8x$. So according to the chain rule, $y' = 7(4x^2 + 1)^6(8x) = 56x(4x^2 + 1)^6$.■

Example 18.2
Find the derivative of $f(x) = \frac{x}{x^2+1}$.

Solution.
We already know one way to find the derivative of this function which is the use of the quotient rule. Another way, is to use the product rule combined with the chain rule since $f(x) = x(x^2 + 1)^{-1}$.

$$\begin{array}{rcl} f'(x) & = & (x)'(x^2+1)^{-1} + x[(x^2+1)^{-1}]' \\ & = & (x^2+1)^{-1} - x(x^2+1)^{-2}(2x) \\ & = & \frac{1}{x^2+1} - \frac{2x^2}{(x^2+1)^2} \end{array}$$ ■

Example 18.3
Prove the power rule for rational exponents.

Solution.
Suppose that $y = x^{\frac{p}{q}}$, where p and q are integers with $q > 0$. Take the qth power of both sides to obtain $y^q = x^p$. Differentiate both sides with respect to x to obtain $qy^{q-1}y' = px^{p-1}$. Thus,

$$y' = \frac{p}{q}\frac{x^{p-1}}{x^{\frac{p(q-1)}{q}}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

Note that we are assuming that x is chosen in such a way that $x^{\frac{p}{q}}$ is defined.■

Example 18.4
Show that $\frac{d}{dx}x^n = nx^{n-1}$ for $x > 0$ and n is any real number.

Solution.
Since $x^n = e^{n\ln x}$ then

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} = e^{n\ln x} \cdot \frac{n}{x} = nx^{n-1}. \blacksquare$$

We end this section by finding the derivative of $f(x) = \ln x$ using the chain rule. Write $y = \ln x$. Then $e^y = x$. Differentiate both sides with respect to x to obtain

$$e^y \cdot y' = 1.$$

Solving for y' we find

$$y' = \frac{1}{e^y} = \frac{1}{x}.$$

Recommended Problems (pp. 149 - 150): 3, 15, 23, 29, 30, 32, 33, 35, 36, 38.

19 The Product and Quotient Rules

At this point we don't have the tools to find the derivative of either the function $f(x) = x^3 e^{x^2}$ or the function $g(x) = \frac{x^2}{e^x}$. Looking closely at the function $f(x)$ we notice that this function is the product of two functions, namely, x^3 and e^{x^2}. On the other hand, the function $g(x)$ is the ratio of two functions. Thus, we hope to have a rule for differentiating a product of two functions and one for differentiating the ratio of two functions.
We start by finding the derivative of the product $u(x)v(x)$, where u and v are differentiable functions:

$$\begin{array}{rcl} (u(x)v(x))' &=& \lim_{h\to 0} \frac{u(x+h)v(x+h)-u(x)v(x)}{h} \\ &=& \lim_{h\to 0} \frac{u(x+h)(v(x+h)-v(x))+v(x)(u(x+h)-u(x))}{h} \\ &=& \lim_{h\to 0} u(x+h) \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} + v(x) \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \\ &=& u(x)v'(x) + u'(x)v(x). \end{array}$$

Note that since u is differentiable so it is continuous and therefore

$$\lim_{h\to 0} u(x+h) = u(x).$$

The formula

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{d}{dx}(v(x)) + \frac{d}{dx}(u(x))v(x). \tag{3}$$

is called the **product rule**.

Example 19.1
Find the derivative of $f(x) = x^3 e^{x^2}$.

Solution.
Let $u(x) = x^3$ and $v(x) = e^{x^2}$. Then $u'(x) = 3x^2$ and $v'(x) = 2xe^{x^2}$. Thus, by the product rule we have

$$f'(x) = x^3(2x)e^{x^2} + 3x^2 e^{x^2} = 2x^4 e^{x^2} + 3x^2 e^{x^2}. \blacksquare$$

The **quotient rule** is obtained from the product rule as follows: Let $f(x) = \frac{u(x)}{v(x)}$. Then $u(x) = f(x)v(x)$. Using the product rule, we find $u'(x) = f(x)v'(x) + f'(x)v(x)$. Solving for $f'(x)$ to obtain

$$f'(x) = \frac{u'(x) - f(x)v'(x)}{v(x)}.$$

Now replace $f(x)$ by $\frac{u(x)}{v(x)}$ to obtain

$$\begin{aligned}\left(\frac{u(x)}{v(x)}\right)' &= \frac{u'(x)-\frac{u(x)}{v(x)}v'(x)}{v(x)}\\ &= \frac{\frac{u'(x)v(x)-u(x)v'(x)}{v(x)}}{v(x)}\\ &= \frac{u'(x)v(x)-u(x)v'(x)}{(v(x))^2}.\end{aligned}$$

Example 19.2
Find the derivative of $g(x) = \frac{x^2}{e^x}$.

Solution.
Let $u(x) = x^2$ and $v(x) = e^x$. Then by the quotient rule we have

$$\begin{aligned}f'(x) &= \frac{(x^2)'e^x - x^2(e^x)'}{(e^x)^2}\\ &= \frac{2xe^x - x^2e^x}{e^{2x}} \blacksquare\end{aligned}$$

Example 19.3
Prove the power rule for integer exponents.

Solution.
In Section 15, we proved the result for positive integers. The result is trivially true when the exponent is zero. So suppose that $y = x^n$ with n a negative integer. Then $y = \frac{1}{x^{-n}}$ where $-n$ is a positive integer. Applying both the quotient rule and the power rule we find

$$y' = \frac{(0)(x^{-n}) - (-nx^{-n-1})}{x^{-2n}} = nx^{n-1}.\blacksquare$$

Recommended Problems (pp. 152 - 3): 3, 5, 12, 22, 25, 27, 33, 34, 35, 38.

20 Derivatives of Periodic Functions

The goal of this section is to find the derivatives of the sine and cosine functions.
We start by finding the derivative of $y = \sin x$. For this purpose, we remind the reader of the following trigonometric identity:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Using the definition of derivative, the above identity and the fact that $\lim_{h\to 0}\frac{\sin h}{h} = 1, \lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ we find

$$\begin{aligned}
y' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} = \cos x
\end{aligned}$$

Now, if $y = \sin u$ where u is a function of x then by the chain rule we have

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}.$$

As a result of this rule and the fact that $\cos x = \sin\left(x+\frac{\pi}{2}\right)$ and $\cos\left(x+\frac{\pi}{2}\right) = -\sin x$ we can obtain the derivative of $\cos x$:

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\sin\left(x+\frac{\pi}{2}\right) = \cos\left(x+\frac{\pi}{2}\right) = -\sin x.$$

If u is a function of x then by the chain rule

$$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}.$$

Example 20.1
Differentiate: (a) $2\sin(3x)$ (b) $\cos(x^2)$ (c) $e^{\sin x}$.

Solution.
(a) $\frac{d}{dx}(2\sin(3x)) = 2\cos(3x)(3x)' = 6\cos(3x)$.
(b) $\frac{d}{dx}(\cos(x^2)) = -\sin(x^2)(x^2)' = -2x\sin(x^2)$.
(c) $\frac{d}{dx}(e^{\sin x}) = e^{\sin x}(\sin x)' = \cos x e^{\sin x}$.■

Recommended Problems (p. 156): 1, 3, 5, 10, 11, 15, 17, 20.

21 Local Maxima and Minima

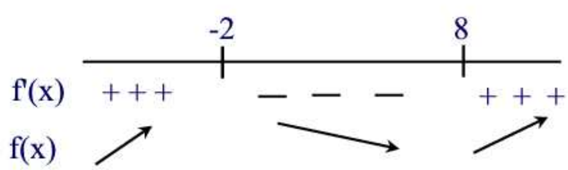
We start this section by reviewing what the first and second derivatives of a function tell us about its graph:

• If $f'(x) > 0$ on an open interval I then $f(x)$ is increasing on I.
• If $f'(x) < 0$ on an open interval I then $f(x)$ is decreasing on I.
• If $f''(x) > 0$ on an open interval I then $f(x)$ is concave up on I.
• If $f''(x) < 0$ on an open interval I then $f(x)$ is concave down on I.

Example 21.1
Consider the function $f(x) = x^3 - 9x^2 - 48x + 52$.

(a) Find the intervals where the function is increasing/decreasing.
(b) Find the intervals where the function is concave up/down.

Solution.
(a) Finding the first derivative we obtain $f'(x) = 3x^2 - 18x - 48 = 3(x-8)(x+2)$. Constructing the chart of signs below



we see that $f(x)$ is increasing on $(-\infty, -2) \cup (8, \infty)$ and decreasing on $(-2, 8)$.
(b) Finding the second derivative, we obtain $f''(x) = 6x - 18 = 6(x-3)$. So, $f(x)$ is concave up on $(3, \infty)$ and concave down on $(-\infty, 3)$.■

Points of interest on the graph of a function are those points that are the highest on the curve, or the lowest, in a specific interval. Such points are called **local extrema.** The highest point, say $f(a)$, is called a **local maximum** and satisfies $f(x) \leq f(a)$ for all x in an interval I. A **local minimum** is a point $f(a)$ such that $f(a) \leq f(x)$ for all x in an interval I containing a.

Example 21.2
Find the local maxima and the local minima of the function given in Figure 34.

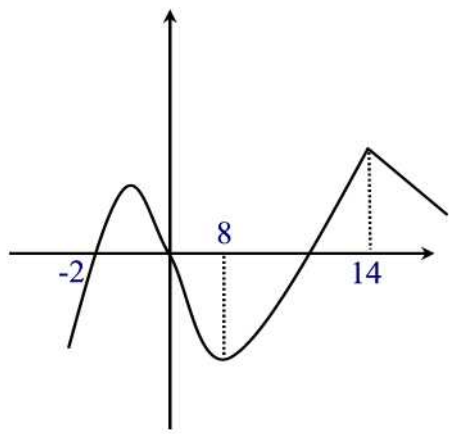


Figure 34

Solution.
The local maxima occur at $x = -2$ and $x = 14$ whereas the local minimum occurs at $x = 8$.■

Next we will discuss two procedures for finding local extrema. We notice from the previous example that local extrema occur at points p where the derivative is either zero or undefined. We call p a **critical number**, $f(p)$ a **critical value**, and $(p, f(p))$ a **critical point**. Most of the critical numbers that we will encounter in this book are of the form $f'(p) = 0$ type. The following theorem asserts that local extrema occur at the critical points.

Theorem 21.1
Suppose that f is defined on an interval I and has a local maximum or minimum at an interior point a. If f is differentiable at a then $f'(a) = 0$.

Remark 21.1
By Theorem 21.1, local extrema are always critical points. The converse of this statement is not true in general. That is, there are critical points that are not local extrema of a function. An example of this situation is given next.

Example 21.3
Show that $f(x) = x^3$ has a critical point at $x = 0$ but 0 is neither a local maximum nor a local minimum.

Solution.
Finding the derivative to obtain $f'(x) = 3x^2$. Setting this to 0 we find the critical point $x = 0$. Since $f'(x)$ does not change sign at 0 then 0 is neither a local maximum nor a local minimum.■

The graph in Example 21.2 suggests two tests for finding local extrema. The first is known as the **first derivative test** and the second as the **second derivative test.**

First-Derivative Test
Suppose that a continuous function f has a critical point at p.
• If f' changes sign from negative to positive at p, then f has a local minimum at p.
• If f' changes sign from positive to negative at p, then f has a local maximum at p. See Figure 35.

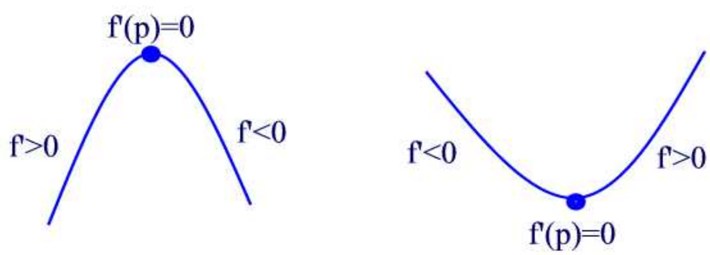


Figure 35

Example 21.4
(a) Find the local extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$.
(b) Find the local extrema of the function $f(x) = \sin x + e^x, x \geq 0$.

Solution.
(a) Using the chart of signs of f' discussed in Example 21.1, we find that $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 8$.
(b) Finding the derivative to obtain $f'(x) = \cos x + e^x$. But for $x \geq 0$,

$1 \leq e^x$. Since $-1 \leq \cos x \leq 1$ then adding the two inequalities we see that $0 \leq \cos x + e^x$. This implies that f' does not change sign for $x \geq 0$. Therefore, there are no local maxima. The only local minimum occurs at $(0, 1)$.■

Second-Derivative Test
Let f be a continuous function such that $f'(p) = 0$.
• if $f''(p) > 0$ then f has a local minimum at p.
• if $f''(p) < 0$ then f has a local maximum at p.
• if $f''(p) = 0$ then the test fails. In this case, it is recommended that you use the first derivative test.

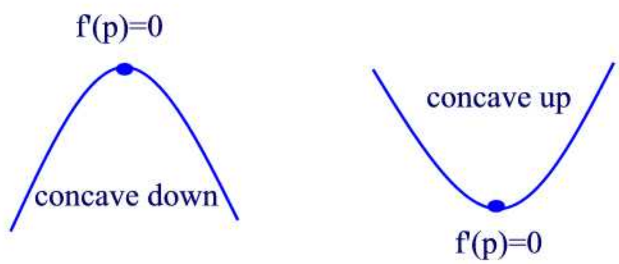


Figure 36

Example 21.5
Use the second derivative test to find the local extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$.

Solution.
The second derivative of $f(x)$ is given by $f''(x) = 6(x - 3)$. The critical numbers are -2 and 8. Since $f''(-2) = -30 < 0$ then $x = -2$ is a local maximum. Since $f''(8) = 30 > 0$ then $x = 8$ is a local minimum.■

Example 21.6
Find the local extrema of the function $f(x) = x^4$.

Solution.
Let's try and find the local extrema by using the second derivative test. Since $f'(x) = 4x^3$ then $x = 0$ is the only critical number. Since $f''(x) = 12x^2$ then $f''(0) = 0$. So the second derivative test is inconclusive. Now, using the first derivative test, we see that $f'(x)$ changes sign from negative to positive at

$x = 0$. Thus, $x = 0$ is a local minimum.$\blacksquare$

Recommended Problems (pp. 170 - 1): 1, 3, 5, 7, 9, 12, 15, 17, 19, 21.

22 Concavity and Points of Inflection

We have seen that a local extremum is a point where the first derivative changes sign. In this section we will discuss points where the second derivative changes sign. That is, the points where the graph of the function changes concavity. We call such points **points of inflection**.
How do you find the points of inflection? Well, since f'' changes sign on the two sides of an inflection point then it makes sense to say that points of inflection occur at points where either the second derivative is 0 or undefined.

Example 22.1
Find the point(s) of inflection of the function $f(x) = xe^{-x}$.

Solution.
Using the product rule to obtain $f'(x) = e^{-x} - xe^{-x}$. Using the product rule for the second time we find $f''(x) = e^{-x}(x-2)$. Thus, a candidate for a point of inflection is $x = 2$. Since $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$ then $x = 2$ is a point of inflection.$\blacksquare$

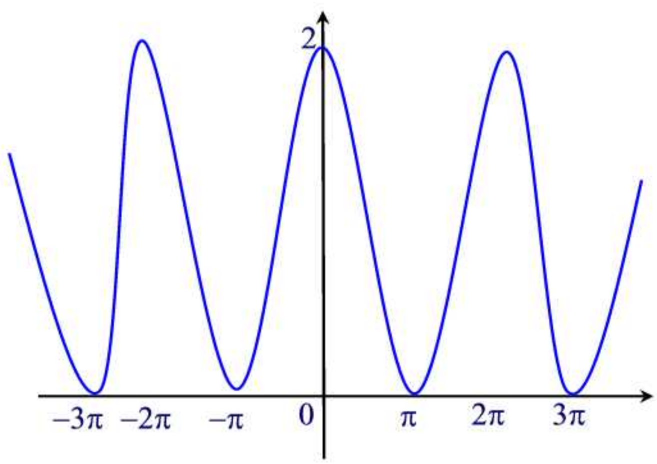
Remark 22.1
We have seen that not every value of x where the derivative is zero or undefined is a local maximum or minimum. The same thing applies for points of inflection. That is, it is not always true that if the second derivative is 0 or undefined then automatically you have a point of inflection. It is critical that f'' changes sign at such a point in order to have a point of inflection.

Example 22.2
Consider the function $f(x) = x^4$. Show that $f''(0) = 0$ but 0 is not a point of inflection.

Solution.
The second derivative is given by the formula $f''(x) = 12x^2$. Clearly, $f''(0) = 0$. Since $f''(x) \geq 0$, that is, $f''(x)$ does not change sign then 0 is not a point of inflection.$\blacksquare$

Example 22.3
Graph the derivative of the function $f(x) = x + \sin x$. Determine where f is increasing most rapidly, and least rapidly.

Solution.
The derivative of $f(x)$ is given by the expression $f'(x) = \cos x + 1 \geq 0$ so that $f(x)$ is always increasing. Now, $f(x)$ increases most rapidly at the maximum values of $f'(x)$ and increases least rapidly at the minimum values of $f'(x)$. Graphing the function $f'(x)$ we find



Thus, f increases most rapidly at $x = 2n\pi$ and least rapidly at $x = (2n+1)\pi$ where n is an integer.■

Example 22.4
Graph a function with the following properties:

- f has a critical point at $x = 4$ and an inflection point at $x = 8$.
- $f' < 0$ for $x < 4$ and $f' > 0$ for $x > 4$.
- $f'' > 0$ for $x < 8$ and $f'' < 0$ for $x > 8$.

Solution.
The graph is given in Figure 37.

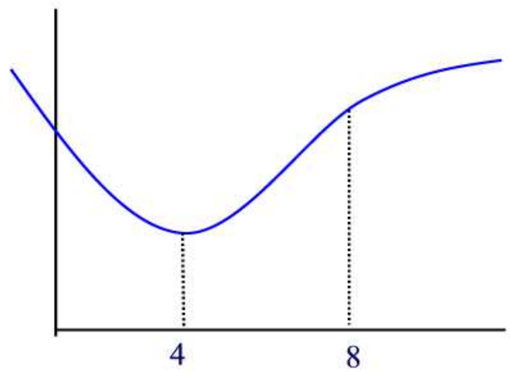


Figure 37

Recommended Problems (pp. 175 - 6): 1, 3, 7, 9, 15, 16, 26, 27, 28, 29.

23 Global Maxima and Minima

In this section we will look for the largest or the smallest values of a function on its domain. Such points are called **global extrema**. If $f(a)$ is the largest value then it satisfies the inequality $f(x) \leq f(a)$ for all x in the domain of f. We call $f(a)$ the **global or absolute maximum value** of f and the point $(a, f(a))$ the global maximum point. Similarly, if $f(a)$ is the smallest value of $f(x)$ then $f(a) \leq f(x)$ for all x in the domain of f. We call $f(a)$ the **absolute or global minimum value** of f and the point $(a, f(a))$ the global minimum.
The process of finding the global extrema is called **optimization**. Problems that involve finding the global extrema are called **optimization problems**.

How do we find the global extrema?
• If the function is continuous on a closed interval then the global extrema occur at either the critical points or the endpoints of the interval.

Example 23.1
Find the global extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$ on the closed interval $[-5, 12]$.

Solution.
Finding the derivative of $f(x)$ we get $f'(x) = 3x^2 - 18x - 48$. Solving the equation $f'(x) = 0$ that is, $x^2 - 6x - 16 = 0$ we find the critical points at $x = 8$ and $x = -2$. Now, evaluating the function at these points and at the endpoints we find

$$\begin{array}{rcl} f(-5) & = & -58 \\ f(-2) & = & 104 \\ f(8) & = & -396 \\ f(12) & = & -92 \end{array}$$

It follows that $(-2, 104)$ is the global maximum point and $(8, -396)$ is the global minimum point.■
• If a function is continuous on an open interval or on all real numbers then it is recommended to find the global extrema by graphing the function.

Example 23.2
Find the global extrema of the function $f(x) = 100(e^{-0.02x} - e^{-0.1x})$ for $x \geq 0$.

Solution.
Let's sketch the graph of this function. The standard process of graphing consists of the following steps:

Step 1. Find the critical numbers. Setting $f'(x) = 0$ to obtain

$$\begin{array}{ccccc}
100(-0.02e^{-0.02x} + 0.1e^{-0.1x}) & = & 0 & & \\
0.02e^{-0.02x} & = & 0.1e^{-0.1x} & & \\
\frac{e^{-0.02x}}{e^{-0.1x}} & = & \frac{0.1}{0.02} & & \\
e^{0.08x} & = & 5 & & \\
0.08x & = & \ln 5 & & \\
x & = & \frac{\ln 5}{0.08} & = & 20.12
\end{array}$$

Step 2. We construct the following chart:

x		20.12	
f'(x)	+	0	-
f(x)	$\nearrow$	53.50	$\searrow$

Step 3. Find the second derivative to obtain $f''(x) = 100(0.0004e^{-0.02x} - 0.01e^{-0.1x})$. Setting this to zero and solving for x as in Step 1 we find $x \approx 40.25$. Now we construct the table

x		40.25	
f''(x)	-	0	+
f(x)	$\cap$	f(40.25)	$\cup$

Step 4.Graph

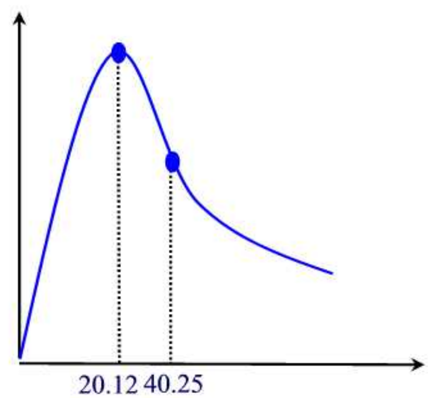


Figure 38

Thus, from the graph we see that $(20.12, 53.50)$ is a global maximum. The function has a global minimum at $x = 0$.■

Recommended Problems (pp. 179 - 182): 2, 3, 5, 9, 11, 13, 19, 21, .

24 Applications of Optimization to Marginality

Management of most businesses always aim to maximizing profit. In this section we will use the derivative to optimizie profit and revenue functions.

Optimizing Profit
Recall that the profit resulting from producing and selling q items is defined by

$$P(q) = R(q) - C(q)$$

where $C(q)$ is the total cost of producing a quantity q and $R(q)$ is the total revenue from selling a quantity q of some good.
To maximize or minimize profit over a closed interval, we optimize the profit function P. We know that global extrema occur at the critical numbers of P or at the endpoints of the interval. Thus, the process of optimization requires finding the critical numbers which are the zeros of the marginal profit function

$$P'(q) = R'(q) - C'(q) = 0$$

where $R'(q)$ is the marginal revenue function and $C'(q)$ is the marginal cost function. Thus, the global maximum or the global minimum of P occurs when

$$MR(q) = MC(q)$$

or at the endpoints of the interval.

Example 24.1
Find the quantity q which maximizes profit given the total revenue and cost functions

$$\begin{aligned} R(q) &= 5q - 0.003q^2 \\ C(q) &= 300 + 1.1q. \end{aligned}$$

where $0 \leq q \leq 800$ units. What production level gives the minimum profit?

Solution.
The profit function is given by

$$P(q) = R(q) - C(q) = -0.003q^2 + 3.9q - 300.$$

The critical numbers of P are the solutions to the equation $P'(q) = 0$. That is,

$$3.9 - 0.006q = 0$$

or $q = 650$ units. Since $P(0) = -\$300, P(800) = \$900$ and $P(650) = \$967.50$ then the maximum profit occurs when $q = 800$ units and the minimum profit(or loss) occurs when $q = 0$, i.e. when there is no production.■

Example 24.2
The total revenue and total cost curves for a product are given in Figure 39.

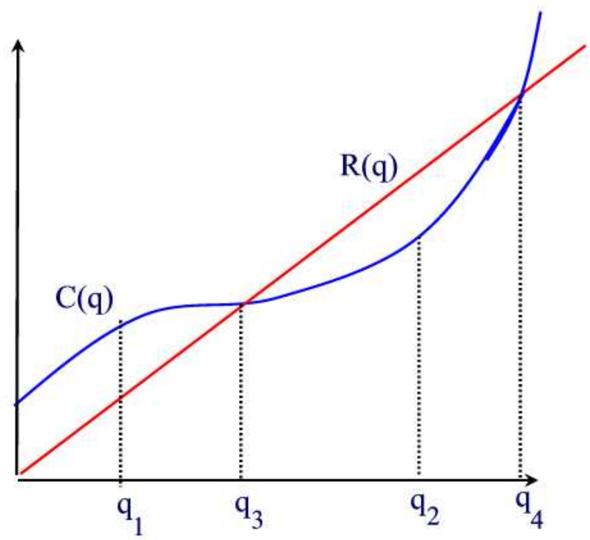


Figure 39

(a) Sketch the curves for the marginal revenue and marginal cost on the same axes. Show on this graph the quantities where marginal revenue equals marginal cost. What is the significance of these two quantities? At which quantity is profit maximum?
(b) Graph the profit function $P(q)$.

Solution.
(a) Since R is a straight line with positive slope then its derivative is a positive constant. That is, the graph of the marginal revenue is a horizontal line at some value $a > 0$. Since C is always increasing then its derivative MC is always positive. For $0 < q < q_3$ the curve is concave down so that MC is decreasing. For $q > q_3$ the graph of C is concave up and so MC is increasing.

Thus, the graphs of C and R are shown in Figure 40.

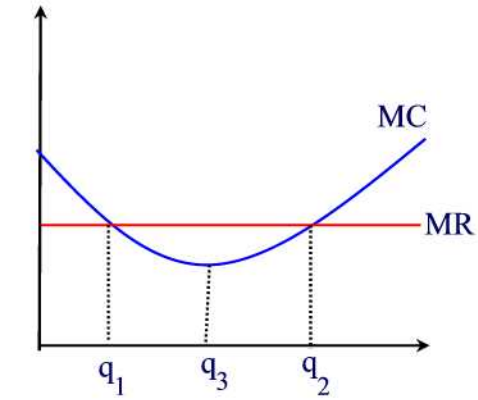


Figure 40

According to the graph, marginal revenue equals marginal cost at the values $q = q_1$ and $q = q_2$. So maximum profit occurs either at q_1, q_2 or at the endpoints. Notice that the production levels q_1 and q_2 correspond to the two points where the tangent line to C is parallel to the tangent line to R. Now, for $0 < q < q_1$ we have $MR < MC$ so that $P' = MR - MC < 0$ and this shows that P is decreasing. For $q_1 < q < q_2$, $MR > MC$ so that $P' > 0$ and hence P is increasing. So P changes from decreasing to increasing at q_1 which means that P has a minimum at q_1. Now, for $q > q_2$ we have that $MR < MC$ so that $P' < 0$ and P is decreasing. Thus, P changes from increasing to decreasing at q_2 so q_2 is a local maximum for P. So maximum profit occurs either at the endpoints or at q_2. Since profit is negative for $q < q_3$ and $q > q_4$ then the profit is maximum for $q = q_2$.
(b)

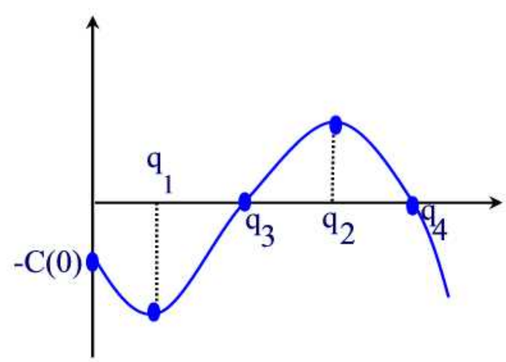


Figure 41

Optimizing Revenue

Example 24.3
The demand equation for a product is $p = 45 - 0.01q$. Write the revenue function as a function of q and find the quantity that maximizes revenue. What price corresponds to this quantity? What is the total revenue at this price?

Solution.
The revenue function is given by $R(q) = pq = 45q - 0.01q^2$. This is a parabola that opens down so its vertex is the global maximum. The maximum then occurs at the critical number of $R(q)$. That is, at the solution of $R'(q) = 0$ or $45 - 0.02q = 0$. Solving for q we find $q = 2250$ units. The maximum revenue is $R(2250) = \$50,625$. The unit price in this case is $p = 45 - 0.01(2250) = \$22.50$ ■

Recommended Problems (pp. 187 - 8): 1, 2, 3, 5, 7, 8, 9, 16, 17.

25 Average Cost

We have seen that an important principle in economics is the problem of maximizing profit. A second general principle involves the relationship between the marginal cost and the **average cost**

$$a(q) = \frac{C(q)}{q}.$$

Example 25.1
The cost of producing q items is $C(q) = 2500 + 12q$ dollars.

(a) What is the marginal cost of producing the 100th item?
(b) What is the average cost of producing 100 items?

Solution.
(a) The marginal cost is given by $MC(q) = 12$. This means that after producing the 99 items, it costs an additional \$12 to produce the 100th item.
(b) $a(100) = \frac{C(100)}{100} = \frac{2500+12(100)}{100} = \$37$ per item.■

Since $a(q) = \frac{C(q)}{q} = \frac{C(q)-0}{q-0}$ then $a(q)$ is the slope of the line passing through the points $(q, C(q))$ and the origin $(0, 0)$. See Figure 42.

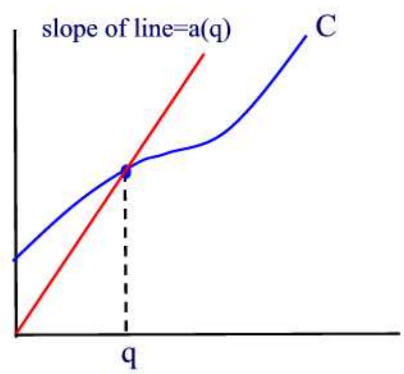


Figure 42

Minimizing $a(q)$
The important question in this section is the question of minimizing the average cost function $a(q)$. Let's try to find the derivative of $a(q)$. Using the

quotient rule of differentiation we obtain

$$a'(q) = \frac{C'(q)q - C(q)}{q^2} = \frac{C'(q) - a(q)}{q}.$$

Thus, $a'(q) = 0$ when $C'(q) = a(q)$. So critical numbers of $a(q)$ satisfy the relationship $C'(q) = a(q)$. In economics theory the global minimum of $a(q)$ occurs at a critical number. Graphically, the minimum average cost occurs at the point on the graph of $C(q)$ where the line passing through the origin is tangent to the graph of $C(q)$. See Figure 43.

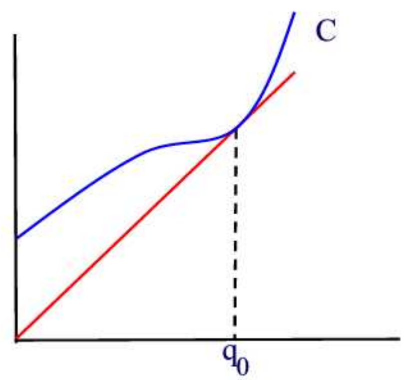


Figure 43

Thus, if q_0 is a critical number of a then for $q < q_0$ the marginal cost is less than the average cost. This means, increasing production will decrease the average cost. If, on the other hand, $q > q_0$ then the marginal cost is greater than the average cost. This means that increasing production will increase the average cost.

Example 25.2
A total cost function, in thousands of dollars, is given by $C(q) = q^3 - 6q^2 + 15q$, where q is in thousands and $0 \leq q \leq 5$.

(a) Graph $C(q)$. Estimate the quantity at which average cost is minimized.
(b) Graph the average cost function. Use it to estimate the minimum average cost.
(c) Determine analytically the exact value of q at which average cost is minimized.

Solution.
(a) A graph of $C(q)$ is given in Figure 44. The average cost is minimized at the point where the line going through the origin is tangent to the graph of $C(q)$. This occur at approximately $q = 3$.

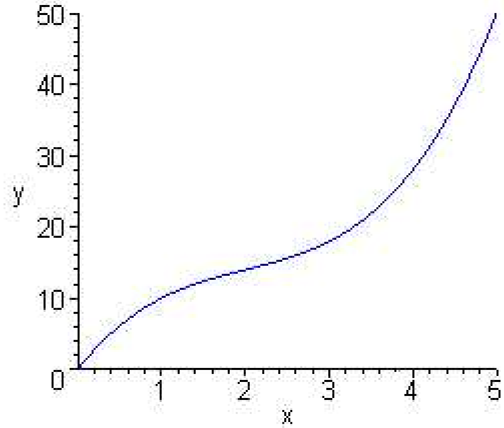


Figure 44

(b) The average cost function is given by $a(q) = \frac{C(q)}{q} = q^2 - 6q + 15$. The graph of this function is given in Figure 45. Notice that the minimum occurs at approximately $q = 3$.

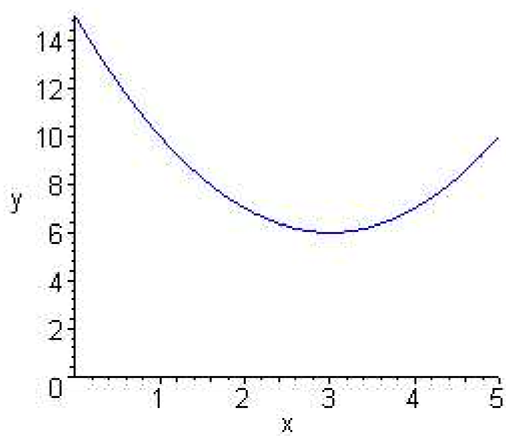


Figure 45

(c) The minimum average cost occurs when $C'(q) = a(q)$. That is, $3q^2 - 12q + 15 = q^2 - 6q + 15$. This gives $2q^2 - 6q = 0$. Solving for q we find $q = 0$ or $q = 3$. Since the average cost is not defined when $q = 0$ then the average cost is minimum at $q = 3$. ■

Recommended Problems (pp. 192 - 3): 1, 3, 5, 9, 11, 13.

26 Elasticity of Demand

An important quantity in economics theory is the **price elasticity of demand** which measures the responsiveness of demand to a given change in price and is found using the formula

$$\begin{aligned} E &= \left|\frac{\textit{percentage change in quantity demanded}}{\textit{percentage change in price}}\right| \\ &= \left|\frac{\frac{dq}{q}}{\frac{dp}{p}}\right| \\ &= \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \end{aligned}$$

Changing the price of an item by 1% causes a change of E% in the quantity sold. If $E > 1$ then this means that an increase (or decrease) of 1% in price causes demand to drop (increase) by more than one percent. In this case, we say that the demand is **elastic.** If $0 \le E < 1$ then an increase (decrease) of 1% in price causes demand to drop (increase) by less than one percent and in this case we say that the demand is **inelastic**.
Note that we are assuming that increasing the price usually decreases demand and decreasing the price will increase demand so that $\frac{dq}{q}$ and $\frac{dp}{p}$ have opposite sign, that is their ratio is always negative.

Example 26.1
Raising the price of hotel rooms from \$75 to \$80 per night reduces weekly sales from 100 rooms to 90 rooms.

(a) What is the elasticity of demand for rooms at a price of \$75?
(b) Should the owner raise the price?

Solution.
(a) The percent change in price is $\frac{\Delta p}{p} = \frac{80-75}{75} = 0.067 = 6.7\%$ and the percent change in demand is $\frac{\Delta q}{q} = \frac{90-100}{100} = -0.1 = -10\%$. Thus, the elasticity of demand is $E = \frac{0.1}{0.067} = 1.5$.
(b) The weekly revenue at the price of \$75 is $100 \cdot 75 = \$7500$ whereas at the price of \$80 the weekly revenue is $90 \cdot 80 = \$7200$. A price increase results in loss of revenue, so the price should not be raised.■

Example 26.2
The demand for a product is $q = 2000 - 5p$ where q is units sold at a price

of p dollars. Find the elasticity if the price is \$10, and interpret your answer in terms of demand.

Solution.
Using Leibniz Notation we find $\left.\frac{dq}{dp}\right|_{p=10} = -5$ and for $p = 10$ the corresponding quantity is $q = 2000 - 50 = 1950$ so that the elasticity is

$$E = \left|\frac{p}{q}\frac{dq}{dp}\right| = \frac{10 \cdot 5}{1950} = 0.03.$$

The demand is inelastic at the given price; a 1% increase in price will result in a decrease of 0.03% in demand.■
Finally, we would like to know the price that maximizes revenue. That is, the price that brings the greatest revenue. Recall that the revenue function is given by $R = pq$ so that $\frac{dR}{dp} = q + p\frac{dq}{dp} = q(1 + \frac{p}{q}\frac{dq}{dp})$.
If $E > 1$ then $\frac{p}{q}\frac{dq}{dp} < -1$ so that $1 + \frac{p}{q}\frac{dq}{dp} < 0$ and therefore $\frac{dR}{dp} < 0$. This says, that increasing price will decrease revenue or decreasing the price will increase revenue. If $E < 1$ then $\frac{p}{q}\frac{dq}{dp} > -1$ so that $1 + \frac{p}{q}\frac{dq}{dp} > 0$ and consequently $\frac{dR}{dp} > 0$. This means that increasing price will increase revenue. Finally, note that if $\frac{dR}{dp} = 0$ then $E = 1$. That is, $E = 1$ at the critical points of the revenue function. See Figure 46.

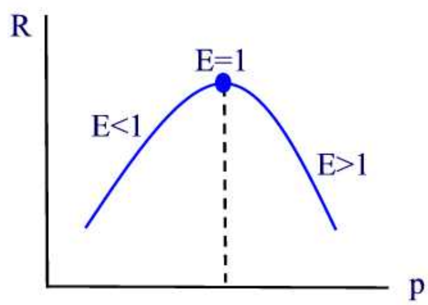


Figure 46

Recommended Problems (pp. 196 - 8): 1, 2, 3, 7, 8, 10, 12, 13, 17, 18.

27 Logistic Functions

One of the consequences of exponential growth is that the output $f(t)$ increases indefinitely in the long run. However, in some situations there is a limit L to how large $f(t)$ can get. For example, the population of bacteria in a laboratory culture, where the food supply is limited. In such situations, the rate of growth slows as the population reaches the carrying capacity. One useful model is the **logistic growth model.**
Thus, logistic functions model *resource-limited* exponential growth.
A **logistic function** involves three positive parameters L, C, k and has the from

$$f(t) = \frac{L}{1 + Ce^{-kt}}.$$

We next investigate the meaning of these parameters. From our knowledge of the graph of e^{-x} we can easily see that $e^{-kt} \to 0$ as $t \to \infty$. Thus, $f(t) \to L$ as $t \to \infty$. It follows that the parameter L represents the limiting value of the output past which the output cannot grow. We call L the **carrying capacity.**
Now, to interpret the meaning of C, we let $t = 0$ in the formula for $f(t)$ and obtain $(1 + C)f(0) = L$. This shows that C is the number of times that the initial output must grow to reach L. Finally, the parameter k affects the steepness of the curve, that is, as k increases, the curve approaches the asymptote $y = L$ more rapidly.

Example 27.1
Show that a logistic function is approximately exponential function with continuous growth rate k for small values of t.

Solution.
Rewriting a logistic function in the form

$$f(t) = \frac{Le^{kt}}{e^{kt} + C}$$

we see that $f(t) \approx \frac{L}{1+C}e^{kt}$ for small values of t.■

Graphs of Logistic Functions
Graphing the logistic function $f(t) = \frac{185}{1+48e^{-0.032t}}$ (See Figure 47) we find

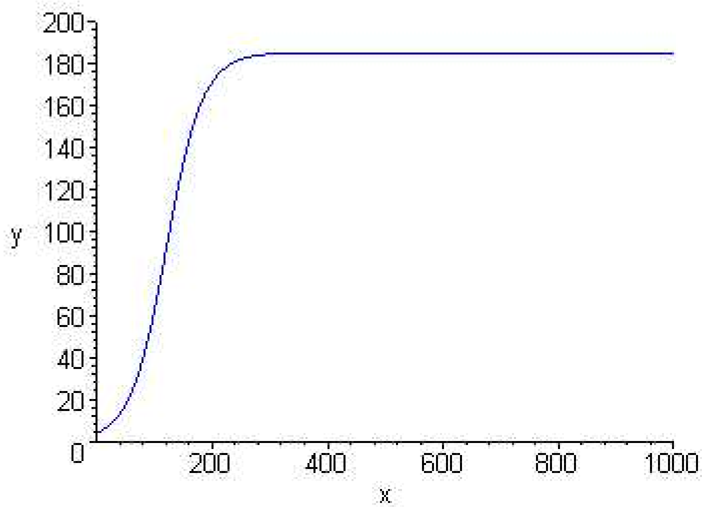


Figure 47

As is clear from the graph above, a logistic function shows that initial exponential growth is followed by a period in which growth slows and then levels off, approaching (but never attaining) a maximum upper limit.
Notice the characteristic S-shape which is typical of logistic functions.

Point of Diminishing Returns
Another important feature of any logistic curve is related to its shape: *every logistic curve has a single inflection point which separates the curve into two equal regions of opposite concavity.* This inflection point is called the **point of diminishing returns.**

Finding the Coordinates of the Point of Diminishing Returns(Optional)
To find the point of inflection of a logistic function of the form $P = f(t) = \frac{L}{1+Ce^{-kt}}$, we notice that P satisfies the equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

Using the product rule we find

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}\left(1 - \frac{2P}{L}\right).$$

Since $\frac{dP}{dt} > 0$ we conclude that $\frac{d^2P}{dt^2} = 0$ at $P = \frac{L}{2}$.
To find y, we set $y = \frac{L}{2}$ and solve for t :

$$\begin{array}{rcl}
\frac{L}{2} & = & \frac{L}{1+Ce^{-kt}} \\
\frac{2}{L} & = & \frac{1+Ce^{-kt}}{L} \\
2 & = & 1 + Ce^{-kt} \\
1 & = & Ce^{-kt} \\
e^{kt} & = & C \\
kt & = & \ln C \\
t & = & \frac{\ln C}{k}
\end{array}$$

Thus, the coordinates of the diminishing point of returns are $\left(\frac{\ln C}{k}, \frac{L}{2}\right)$.

Logistic functions are good models of biological population growth in species which have grown so large that they are near to saturating their ecosystems, or of the spread of information within societies. They are also common in marketing, where they chart the sales of new products over time.

Example 27.2
The following table shows that results of a study by the United Nations (New York Times, November 17, 1995) which has found that world population growth is slowing. It indicates the year in which world population has reached a given value:

Year	1927	1960	1974	1987	1999	2011	2025	2041	2071
Billion	2	3	4	5	6	7	8	9	10

(a) Construct a scatterplot of the data, using the input variable t is the number of years since 1900 and output variable $P =$ worldpopulation (in billions).
(b) Using a logistic regression, fit a logistic function to this data.
(c) Find the point of diminishing returns. Interpret its meaning.

Solution.
(a) For this part, we recommand the reader to use a TI for the plot.
(b) Using a TI with the logisitc regression we find $L = 11.5, C = 12.8, k = 0.0266$. Thus,

$$P = \frac{11.5}{1 + 12.8e^{-0.0266t}}.$$

(c) The inflection point on the world population curve occurs when $t = \frac{\ln C}{k} = \frac{\ln 12.8}{0.0266} \approx 95.8$. In other words, according to the model, in 1995 world population attained 5.75 billion, half its limiting value of 11.5 billion. From this year on, population will continue to increase but at a slower and slower rate. ■

Recommended Problems (pp. 204 - 6): 3, 4, 5, 7.

28 Measuring The Distance Traveled

We have seen that the velocity of an object moving along the curve $s(t)$ is obtained by taking the average rate of change on smaller and samller intervals, that is finding the derivative of s, i.e. $v(t) = s'(t)$. In this and the following sections we want to go the opposite direction. That is, given the velocity function $v(t)$ we want to find the position function $s(t)$.
To be more precise, suppose that we want to estimate the distance s traveled by a car after 10 seconds of departure. Assume for example, that we are given the velocity of the car every two seconds as shown in the table below

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50

Since we don't know the instantaneous velocity of the car at every momentm then we can not calculate the distance exactly. What we can do is to estimate the distance traveled. For the first two seconds, the velocity is at least 20 miles per second so that the distance traveled is at least $20 \times 2 = 40$ feet. Likewise, at least $30 \times 2 = 60$ feet has been traveled the next two seconds and so on. Thus, we obtain a lower estimate to the exact distance traveled

$$20 \times 2 + 30 \times 2 + 38 \times 2 + 44 \times 2 + 48 \times 2 = 360 \ feet.$$

However, we can reason differently and get an overestimate to the total distance traveled as follows: For the first two seconds the car's velocity is at most 30 feet so that the car travels at most $30 \times 2 = 60$ feet. In the next two seconds, it travels $38 \times 2 = 76$ feet and so on. So an upper estimate of the total distance traveled is

$$30 \times 2 + 38 \times 2 + 44 \times 2 + 48 \times 2 + 50 \times 2 = 420 \ feet$$

Hence,

$$360 \ feet \leq \ Total \ distance \ traveled \leq 420 \ feet.$$

Notice that the difference between the upper and lower estimates is 60 feet. Figure 48 shows both the lower estimate and the upper estimate. The graph of the velocity is obtained by plotting the points given in the above table and then connect them with a smooth curve. The area of the lower rectangles represent the lower estimate and the larger rectangles represent the upper estimate.

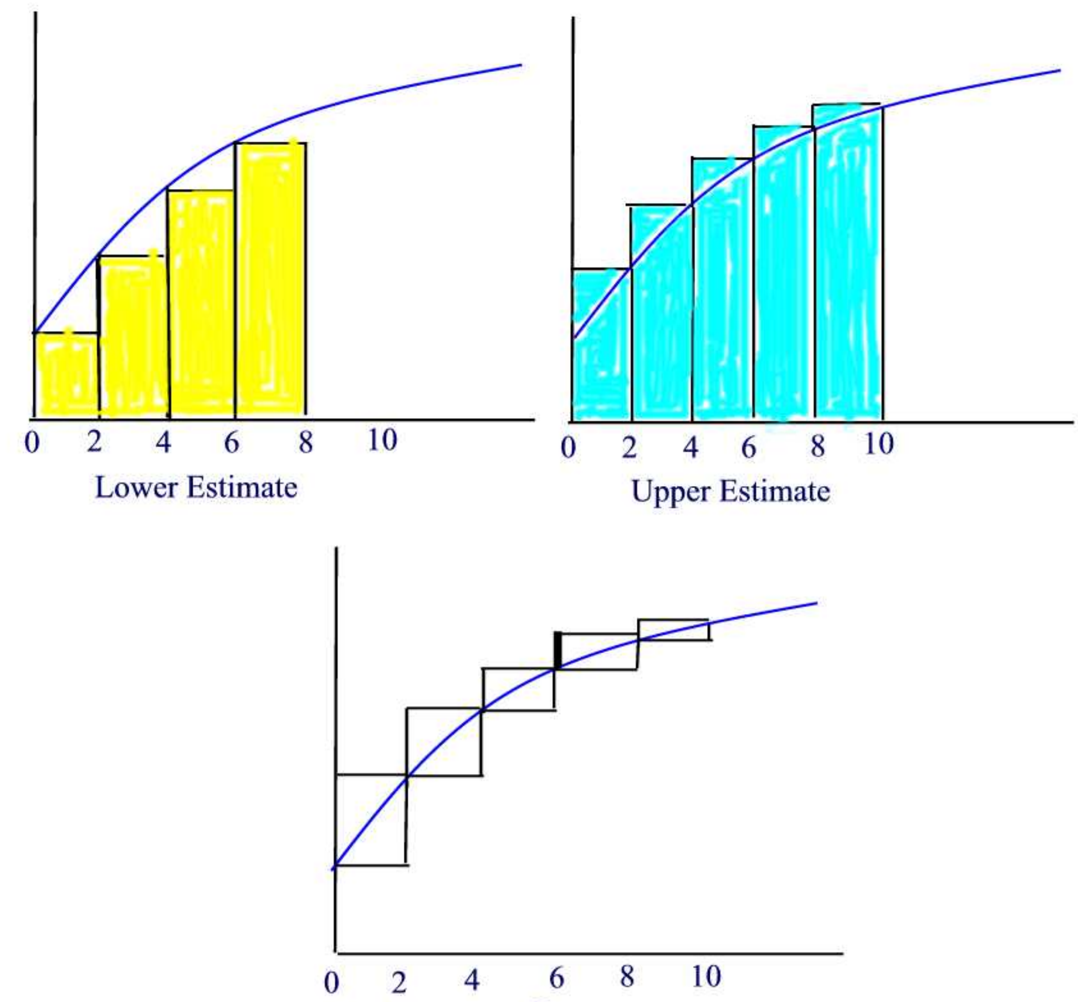


Figure 48

To visualize the difference between the upper and lower estimates, look at the above figure, and imagine that all the unshaded rectangles are pushed to the right and stacked on top of each other. This gives a rectangle of width 2 and height 30 so its area is the difference between the estimates.

Example 28.1
Suppose that the velocity of the car is given every second instead as shown in the table below. Find the lower and upper estimates of the total distance

traveled. What is the difference between the lower and upper estimates? Do you think that knowing the velocity at every second is a better estimate than knowing the velocity every two seconds?

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	35	38	42	44	46	48	49	50

Solution.
The lower estimate is

$$(20)(1) + (26)(1) + \cdots + (48)(1) + (49)(1) = 378\ feet$$

and the upper estimate is

$$(26)(1) + (30)(1) + \cdots + (49)(1) + (50)(1) = 408\ feet$$

Hence,

$$378\ feet \leq\ Total\ distance\ traveled \leq 408\ feet.$$

So the difference between the upper and lower estimates is $408 - 378 = 30\ feet$. This shows that by increasing the partition points we get better and better estimate.■

Remark 28.1
Once the upper estimate and the lower estimate are found then one can get an even better estimate by taking the average of the two estimates.

The use of the average rate of change of the distance leads to finding the total distance traveled. This same method can be used to find the total change from the rate of change of other quantities.

Example 28.2
The following table gives world oil consumptions, in billions of barrels per year. Estimate the total oil consumption during this 20-year period.

Year	1980	1985	1990	1995	2000
Oil (barrels/yr)	22.3	23.0	23.9	24.9	27.0

Solution.
We underestimate the total oil consumption as follows:

$$22.3 \times 5 + 23.0 \times 5 + 23.9 \times 5 + 24.9 \times 5 = 470.5 \textit{ billion barrels}.$$

The overestimate is

$$23.0 \times 5 + 23.9 \times 5 + 24.9 \times 5 + 27.0 \times 5 = 494 \textit{ billion barrels}.$$

A good single estimate of the total oil consumption is the average of the above estimates. That is

$$Total\ oil\ consumption \approx \frac{470.5 + 494}{2} = 482.25 \textit{ billion barrels}.\blacksquare$$

Recommended Problems (pp. 223 - 5): 1, 3, 7, 9, 10, 13, 18.

29 The Definite Integral

In the previous section, we saw how to approximate total change given the rate of change. In this section we see how to make the approximation more accurate.

Suppose that we want to find the total distance traveled over the time interval $a \leq t \leq b$. We take measurements of the velocity $v(t)$ at equally spaced times, $a = t_0, t_1, t_2, \cdots, t_n = b$. This means that we devide the interval $[a, b]$ into n equal pieces each of length $\Delta t = \frac{b-a}{n}$. We first use the left-end point of each interval $[t_{i-1}, t_i]$ and construct the **left-hand sum**

$$L(v, n) = v(t_0)\Delta t + v(t_1)\Delta t + \cdots + v(t_{n-1})\Delta t.$$

Geometrically, this sum represents the sum of areas of rectangles constructed by taking the height to be the value of the function at the left-endpoint of each subinterval. See Figure 49.

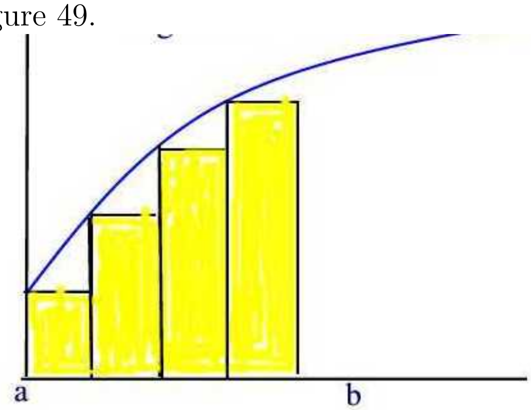


Figure 49

Secondly, we use the right-end point of each interval $[t_{i-1}, t_i]$ and construct the **right-hand sum**

$$R(v, n) = v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t.$$

Geometrically, this sum represents the sum of areas of rectangles constructed by taking the height to be the value of the function at the right-endpoint of each subinterval. See Figure 50.

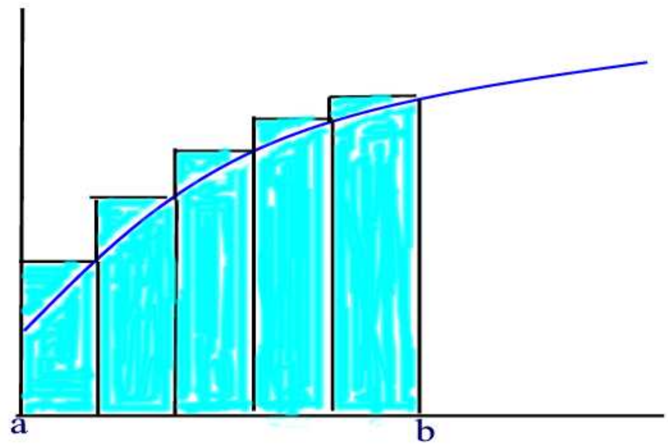
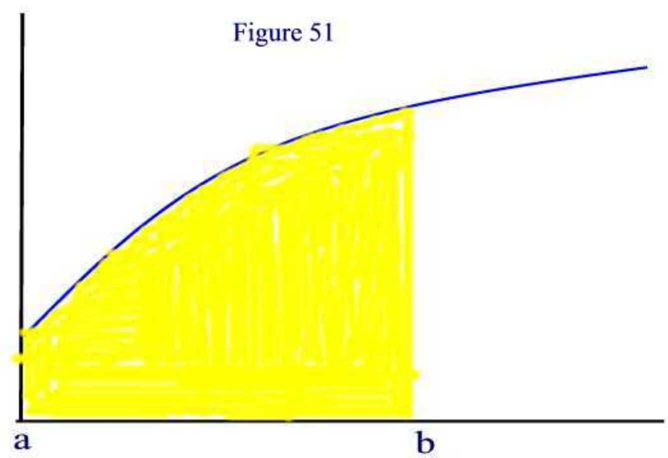


Figure 50

Now, the exact distance traveled lies between the two estimates. As we have seen earlier, by making the time interval smaller and smaller we can make the difference between the two estimates as small as we like. This is equivalent to letting $n \to \infty$. If the function $v(t)$ is continuous then the following two limits are equal to the exact distance traveled from $t = a$ to $t = b$.

$$Total\ distance\ traveled = \lim_{n\to\infty} L(v,n) = \lim_{n\to\infty} R(v,n).$$

Geometrically, each of the above limit represents the area under the graph of $v(t)$ bounded by the lines $t = a, t = b$ and the horizontal axis. See Figure 51.



Remark 29.1
Notice that for an increasing function the left-hand sum is an underestimate whereas the right-hand sum is an overestimate. This role is reversed for a decreasing function.

The above discussion applies to any continuous function f on a closed interval $[a,b]$. We start by dividing the interval [a,b] into n subintervals each of length

$$\Delta x = \frac{b-a}{n}.$$

Let $a = x_0, x_1, \cdots, x_{n-1}, x_n = b$ be the endpoints of the subdivisions. We construct the **left-hand sum** or the **left Riemann sum**

$$L(f,n) = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

and the **right-hand sum** or the **right Riemann sum**

$$R(f,n) = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

It is shown in advanced calculus that for a continuous function on a closed interval $[a,b]$ that as $n \to \infty$ both the left-hand sum and the right-hand sum exist and are equal. We denote the common value by the notation $\int_a^b f(x)dx$. Thus,

$$\begin{aligned} \int_a^b f(x)dx &= \lim_{n\to\infty} L(f,n) \\ &= \lim_{n\to\infty} R(f,n) \end{aligned}$$

We call $\int_a^b f(x)dx$ the **definite integral** of f from $x = a$ to $x = b$. We call a the **lower limit** and b the **upper limit**. The function f is called the **integrand**.

Example 29.1
(a) On a sketch of $y = \ln x$, represent the left Riemann sum with $n = 2$ approximating $\int_1^2 \ln x dx$. Write out the terms in the sum, but do not evaluate.
(b) On a another sketch of $y = \ln x$, represent the right Riemann sum with $n = 2$ approximating $\int_1^2 \ln x dx$. Write out the terms in the sum, but do not evaluate.
(c) Which sum is an underestimate? Which sum is an overestimate?

Solution.
(a) The left Riemann sum is the sum

$$L(\ln x, 2) = \ln 1(0.5) + \ln(1.5)(0.5) = 0.5\ln(1.5).$$

The sum is represented by the rectangle shaded to the left of Figure 52.

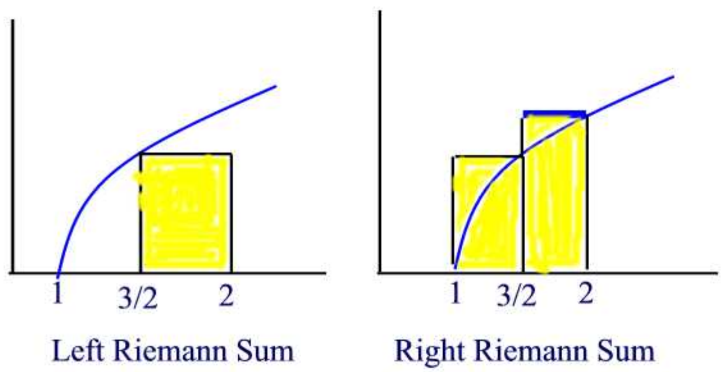


Figure 52

(b) The right Riemann sum is the sum

$$R(\ln x, 2) = \ln(1.5)(0.5) + \ln 2(0.5) = 0.5\ln(2)(1.5) = 0.5\ln 3.$$

The sum is represented by the rectangles shaded to the right of Figure 52.
(c) $L(\ln x, 2) < \int_1^2 \ln x dx < R(\ln x, 2)$.■

Definite integrals are used to find areas. That is, a definite integral is the area under the graph of a function. We will discuss this concept in the next section.

Example 29.2 *(Estimating a Definite Integral from a Table)*
Use the table to estimate $\int_0^{40} f(x)dx$. What values of n and Δx did you use?

x	0	10	20	30	40
f(x)	350	410	435	450	460

Solution.
The values of $f(x)$ are spaces 10 units apart so that $\Delta x = 10$ and $n = \frac{b-a}{\Delta x} = \frac{40-0}{10} = 4$. Calculating the left-hand sum and right-hand sum to obtain

$$\begin{array}{rcl} L(f,4) &=& 350\cdot 10 + 410\cdot 10 + 435\cdot 10 + 450\cdot 10 = 16,450 \\ R(f,4) &=& 410\cdot 10 + 435\cdot 10 + 450\cdot 10 + 460\cdot 10 = 17,550. \end{array}$$

Thus,

$$\int_0^{40} f(x)dx \approx \frac{16,450 + 17,550}{2} = 17,000. \blacksquare$$

Example 29.3 *(Estimating a Definite Integral from a Graph)*
The graph of $f(x)$ is given in Figure 53.

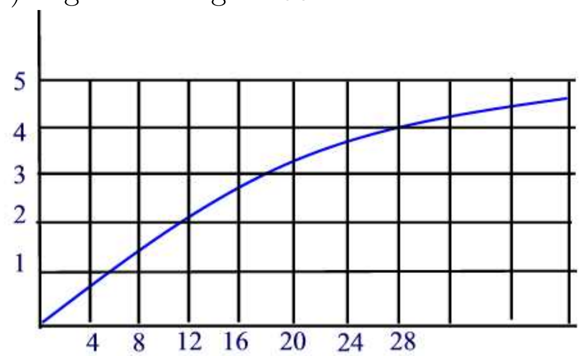


Figure 53

Estimate $\int_0^{28} f(x)dx$.

Solution.
We approximate the integral using left and right-hand sums with $n = 7$ and $\Delta x = 4$.

$$\begin{array}{rcl} L(f,7) & = & 0.75 \cdot 4 + 1.4 \cdot 4 + 2.1 \cdot 4 + 2.8 \cdot 4 + 3.3 \cdot 4 + 3.8 \cdot 4 = 56.6 \\ R(f,7) & = & 1.4 \cdot 4 + 2.1 \cdot 4 + 2.8 \cdot 4 + 3.3 \cdot 4 + 3.8 \cdot 4 + 4 \cdot 4 = 69.6. \end{array}$$

Thus,

$$\int_0^{28} f(x)dx \approx \frac{56.6 + 69.6}{2} = 63.1. \blacksquare$$

Recommended Problems (pp. 231 - 2): 1, 3, 5, 7, 9, 11, 15, 19, 23.

30 The Definite Integral as Area

In this section, we will see how definite integrals are used to find areas.

Case 1: $f(x) \geq 0$

Looking closely to either the left Riemann sum or the right Riemann sum we see that if $f(x) \geq 0$ then a term of the form $f(x)\Delta x$ represents the area of a rectangle. As n increases without bound, that is, the width Δx of the rectangles approaches zero, the rectangles fit the curve of the graph more exactly, and the sum of their areas gets closer and closer to the area under the graph, bounded by the vertical lines $x = a$ and $x = b$ and the x-axis. Thus,

$$\int_a^b f(x)dx = \textit{Area under graph of } f \textit{ between } a \textit{ and } b.$$

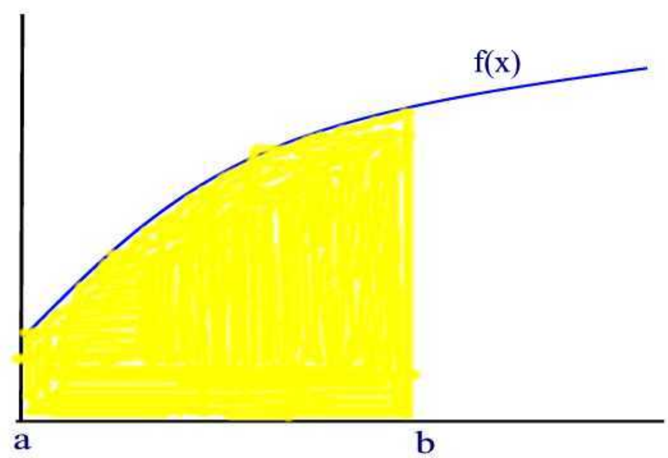


Figure 54

Example 30.1

Consider the integral $\int_{-1}^{1} \sqrt{1-x^2}dx$.

(a) Interpret the integral as an area, and find its exact value.
(b) Estimate the integral using a calculator.

Solution.

(a) Note that the equation of a circle centered at the origin and with radius 1 is given by $x^2 + y^2 = 1$. Solving for y we find $y = \pm\sqrt{1-x^2}$. The function $y = \sqrt{1-x^2}$ corresponds to the upper semicircle and the function $y = -\sqrt{1-x^2}$ corresponds to the lower semicircle. See Figure 55.

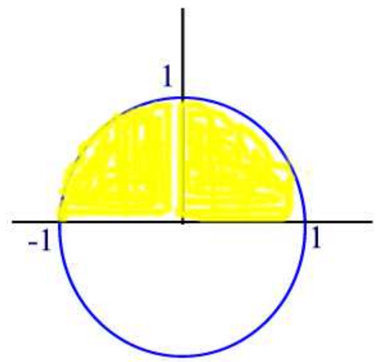


Figure 55

It follows that the given integral represents the area of the upper semicircle and therefore is equal to $\frac{\pi}{2}$. That is,

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{\pi}{2}.$$

(b) Using a TI-83 calculator we find

$$fnInt(\sqrt{1-x^2}, x, -1, 1) \approx 1.571. \blacksquare$$

Case 2: $f(x) \leq 0$

In this case, since each product of the form $f(x)\Delta x$ is less than or equal to zero then the area gets counted negatively. That is, the absolute value of the integral gives the area above the curve between $x = a$ and $x = b$.

Example 30.2

Find the area above the graph of $y = x^2 - 1$ from $x = -1$ to $x = 1$.

Solution.

The graph of $y = x^2 - 1$ is shown in Figure 56.

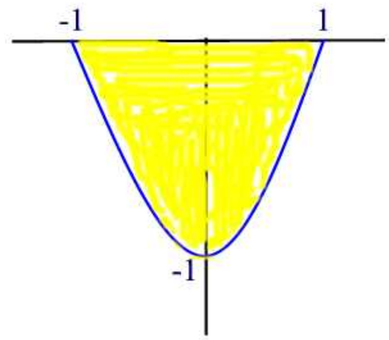


Figure 56

The area is given by $|\int_{-1}^{1}(x^2-1)dx| \approx |-1.33| = 1.33.\blacksquare$

<u>**Case 3:**</u>f changes sign
In this case, the integral is the sum of the areas above the x-axis, counted positively, and the areas below the x-axis, counted negatively. If the integral is positive then the region above the x-axis has larger area than the region below the x-axis. If the integral is negative then the region below the x-axis has a larger area then the region above the x-axis.

Example 30.3
Find the area between the graph of $y = x^3$ and the x-axis from $x = -1$ to $x = 1$.

Solution.
The area is shown in Figure 57.

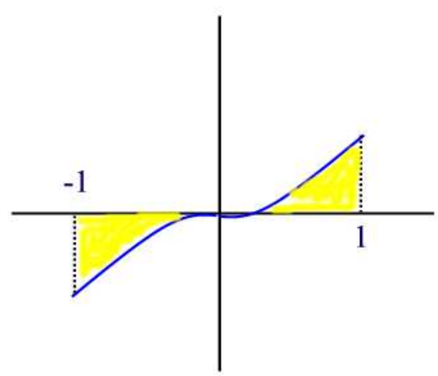


Figure 57

It follows that the area is given by

$$\left|\int_{-1}^{0} x^3 dx\right| + \int_0^1 x^3 dx = 0.5.\blacksquare$$

Area Between Two Curves

Consider the problem of finding the area between two curves as shown in Figure 58.

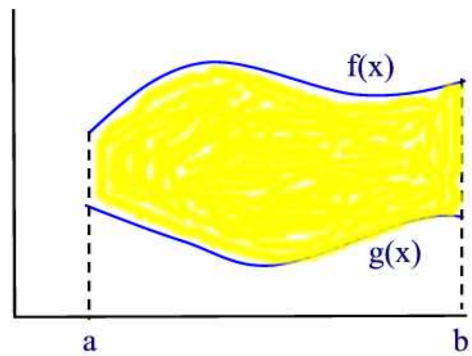


Figure 58

Then, the area between the two curves is the area under f minus the area under g. That is,

$$\textit{Area between } f \textit{ and } g = \int_a^b (f(x) - g(x))dx.$$

Example 30.4

Find the area between the graphs of $f(x) = x$ and $g(x) = x^2$.

Solution.

The area is shown in Figure 59.

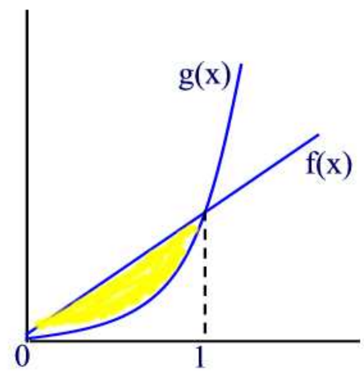


Figure 59

Thus, the area is given by the integral

$$\int_0^1 (x - x^2)dx = \left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = \frac{1}{6}.\blacksquare$$

Recommended Problems (pp. 235 - 7): 1, 3, 5, 7, 11, 13, 17, 19, 22, 25. 27, 30, 31, 32.

31 Interpretations of the Definite Integral

We start this section by showing that the definite integral of a rate of change gives the total change of the function. We define the total change of a function $F(t)$ from $t = a$ to $t = b$ to be the difference $F(b) - F(a)$. Suppose that $F(t)$ is continuous in [a,b] and differentiable in (a, b). Divide the interval $[a, b]$ into n equal subintervals each of length $\Delta t = \frac{b-a}{n}$. Let $a = t_0, t_1, \cdots, t_n = b$ be the partition points of the subdivision. Then on the interval $[t_0, t_1]$ the change in F can be estimated by the formula

$$F'(t_0) \approx \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t}$$

or

$$F(t_0 + \Delta t) - F(t_0) \approx F'(t_0)\Delta t$$

that is

$$F(t_1) - F(t_0) \approx F'(t_0)\Delta t$$

On the interval $[t_1, t_2]$ we get the estimation

$$F(t_2) - F(t_1) = F'(t_1)\Delta t$$

Continuing in this fashion we find that on the interval $[t_{n-1}, t_n]$ we have

$$F(t_{n-1}) - F(t_n) \approx F'(t_{n-1})\Delta t.$$

Adding all these approximations we find that

$$F(t_n) - F(t_0) \approx \sum_{i=0}^{n-1} F'(t_i)\Delta t$$

Letting $n \to \infty$ we see that

$$F(b) - F(a) = \int_a^b F'(t)dt.$$

Example 31.1
The amount of waste a company produces, W, in metric tons per week, is approximated by $W = 3.75e^{-0.008t}$, where t is in weeks since January 1, 2000. Waste removal for the company costs \$15/ton. How much does the company pay for waste removal during the year 2000?

Solution.
The amount of tons produced during the year 2000 is just the definite integral $\int_0^{52} W(t)dt$. Using a calculator we find that

$$Total\ waste\ during\ the\ year = \int_0^{52} 3.75e^{-0.008t}dt \approx 159\ tons$$

The cost to remove this quantity is $159 \times 15 = \$2385.\blacksquare$

Remark 31.1
When using $\int_a^b f(x)dx$ in applications then its units is the product of the units of $f(x)$ with the units of x.

Recommended Problems (pp. 240 - 3): 1, 3, 5, 8, 9, 15, 18.

32 The Fundamental Theorem of Calculus

The following result is considered among the most important result in calculus.

The Fundamental Theorem of Calculus
If $f(x)$ is a continuous function on [a,b] and $F'(x) = f(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

We call the function $F(x)$ an **antiderivative** of $f(x)$.

Proof.
Partition the interval [a,b] into n subintervals each of length $\Delta x = \frac{b-a}{n}$ and let $\{a = x_0, x_1, \cdots, x_n = b\}$ be the partition points. Applying the Mean Value Theorem on the interval $[x_0, x_1]$ we can find a number $x_0 < c_1 < x_1$ such that

$$F(x_1) - F(x_0) = F'(c_1)\Delta x.$$

Continuing this process on the remaining intervals we find

$$\begin{array}{rcl} F(x_2) - F(x_1) & = & F'(c_1)\Delta x \\ & \vdots & \\ F(x_n) - F(x_{n-1}) & = & F'(c_n)\Delta x \end{array}$$

Adding these equalities we find

$$F(x_n) - F(x_0) = \sum_{i=1}^{n} f(c_i)\Delta x$$

Letting $n \to \infty$ to obtain

$$F(b) - F(a) = \int_a^b f(x)dx \blacksquare$$

Example 32.1
Use FTC to compute $\int_1^2 2xdx$. Use a calculator to find the answer to the integral and compare.

Solution.
Since the derivative of x^2 is 2x then $F(x) = x^2$. Thus, by the FTC we have

$$\int_1^2 2xdx = F(2) - F(1) = 4 - 1 = 3.$$

Using a calculator we find $\int_1^2 2xdx = 3.$■

Example 32.2
Let $F(t)$ represent a bacteria population which is 5 million at time $t = 0$. After t hours, the population is growing at an instantaneous rate of 2^t million bacteria per hour. Estimate the total increase in the bacteria population during the first hour, and the population at $t = 1$.

Solution.
Since total change is the definite integral of $F'(t) = 2^t$ from $t = 0$ to $t = 1$ then

$$Change\ in\ population = F(1) - F(0) = \int_0^1 2^t dt \approx 1.44\ million\ bacteria$$

Since $F(0) = 5$ then

$$F(1) = F(0) + \int_0^1 2^t dt \approx 5 + 1.44 = 6.44\ million.$$

If $C(q)$ is the total cost to produce a quantity q of a certain comodity then we can use the Fundamental Theorem of Calculus and compute the total cost of producing b units as follows

$$C(b) - C(0) = \int_0^b C'(q)dq$$

or

$$C(b) = C(0) + \int_0^b C'(q)dq$$

We call the quantity $\int_0^b C'(q)dq$ the **total variable cost.**■

Example 32.3
The marginal cost function for a company is given by

$$C'(q) = q^2 - 16q + 70\ dollars/unit,$$

where q is the quantity produced. If $C(0) = 500$, find the total cost of producing 20 units. What is the fixed cost and what is the total variable cost for this quantity?

Solution.
We find $C(20)$ as follows:

$$C(20) = C(0) + \int_0^{20} C'(q)dq = 500 + \int_0^{20} (q^2 - 16q + 70)dq.$$

where $C(0) = 500$ is the fixed cost.
Using a calculator we find total variable cost to be

$$\int_0^{20} (q^2 - 16q + 70)dq \approx 866.7$$

Thus, the total cost of producing 20 units is

$$C(20) \approx 500 + 866.7 = 1366.7 \blacksquare$$

Recommended Problems (pp. 245 - 6): 2, 3, 5, 7, 9, 11, 16, 17, 20, 25.

33 The Average Value

We know that the average of n given numbers is just the sum divided by n. What is the average in the continuous case? That is, what is the average of a continuous function on a closed interval $[a,b]$?
Partition the interval into n equal subintervals each of length $\Delta t = \frac{b-a}{n}$ and let $a = t_0, t_1, t_2, \cdots, t_n$ be the division points. Then

$$Average\ of\ f(t)\ on\ [a,b] \approx \frac{f(t_0) + f(t_1) + \cdots + f(t_{n-1})}{n}$$

But $n = \frac{b-a}{\Delta t}$ so that

$$\begin{array}{rcl} Average\ of\ f(t)\ on\ [a,b] & \approx & \frac{1}{b-a}(f(t_0) + f(t_1) + \cdots + f(t_{n-1}))\Delta t \\ & = & \frac{1}{b-a}\sum_{i=0}^{n-1} f(t_i)\Delta t \end{array}$$

Letting $n \to \infty$ we see that

$$Average\ of\ f(t)\ on\ [a,b] = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 33.1
A bar of metal is cooling from $1000°C$ to room temperature, $20°C$. The temperature, H, of the bar t minutes after it starts cooling is given by

$$H = 20 + 980e^{-0.1t}.$$

Find the average temperature over the first hour.

Solution.
The average temperature is given by

$$Average\ temperature\ for\ the\ first\ hour = \frac{1}{60}\int_0^{60}(20+980e^{-0.1t})dt \approx 183°C. \blacksquare$$

Example 33.2
Suppose that $C(t)$ represent the daily cost of heating your house, in dollars per day, where t is time in days and $t = 0$ corresponds to January 1, 2002. Interpret the quantities $\int_0^{90} C(t)dt$ and $\frac{1}{90-0}\int_0^{90} C(t)dt$.

Solution.
The integral $\int_0^{90} C(t)dt$ represents the total cost in dollars to heat your house for the first 90 days of 2002. The second expression represents the average cost per day to heat your house during the first 90 days of 2002.$\blacksquare$

Remark 33.1
From the definition of the average value we can write

$$(average\ value\ of\ f) \times (b-a) = \int_a^b f(x)dx$$

Geometrically, this says that the area of the rectangle of dimensions $(average\ value\ of\ f) \times (b-a)$ is equal to the area under the graph of $f(x)$ from $x = a$ to $x = b$. See Figure 60.

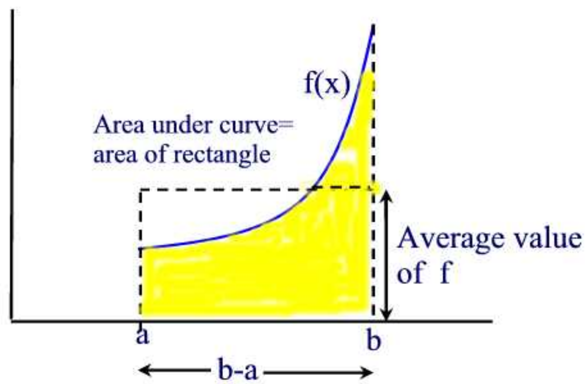


Figure 60

Recommended Problems (pp. 258 - 9): 1, 4, 5, 9, 11, 14, 18.

34 Consumer and Producer Surplus

The definitions of demand and supply must be remembered:

Demand tells us the price that consumers would be willing to pay for each different quantity. According to the law of demand, when the price increases the demand decreases and when the price decreases the demand increases. The graphical representation of the relationship between the quantity demanded of a good and the price of the good is known as the **demand curve.**
Supply tells us the price that producers would be willing to charge in order to sell the different quantities. The law of supply asserts that as the price of a good rises, teh quantity supplied rises, and as the price of a good falls the quantity supplied falls. The graphical representation of the relationship between the quantity supplied of a good and the price of the good is known as the **supply curve.**
The demand and supply curve intersects at the **point of equilibrium** (q^*, p^*). We call p^* the **equilibrium price** and the q^* the **equilibrium quantity.** See Figure 61.

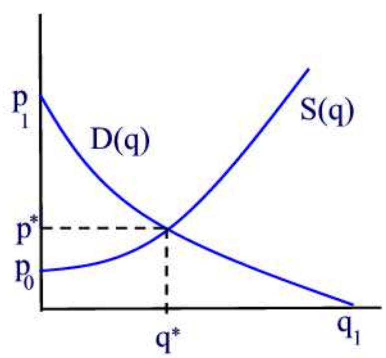


Figure 61

Consumers' Surplus

At the equilibrium level, the **consumers' surplus** is the difference between what consumers are willing to pay and their actual expenditure: It therefore represents the total amount saved by consumers who were willing to pay more than p^* per unit.

To calculate the consumers' surplus, we first calculate the consumers' total expenditure. Divide the interval $[0, q^*]$ into n equal pieces each of length Δq. According to Figure 62, the consumers' total expenditure is given by the sum

$$D(q_1)\Delta q + D(q_2)\Delta q + \cdots + D(q_n)\Delta q = \sum_{i=1}^{n} D(q_i)\Delta q.$$

Letting $\Delta q \to 0$ to obtain (See Figure 62)

$$Total\ Expenditure = \int_0^{q^*} D(q)dq.$$

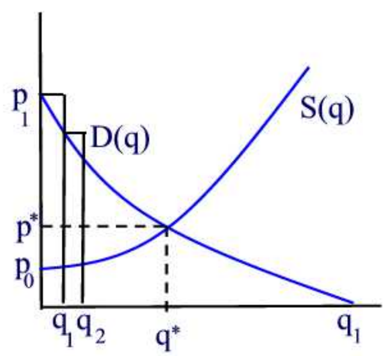


Figure 62

Thus,

$$Consumers'\ Surplus = \int_0^{q^*} (D(q) - p^*)dq.$$

Graphically, it is the area between demand curve and the horizontal line at p^*. See Figure 63.

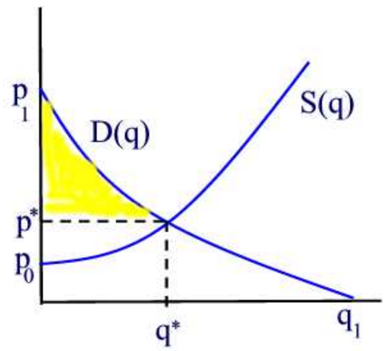


Figure 63

Producers' Surplus
The producers' surplus is the extra amount earned by producers who were willing to charge less than the selling price of p^* per unit, and is given by

$$Producers'\ Surplus = \int_0^{q^*} (p^* - S(q))dq.$$

Graphically, it is the area between suuply curve and the horizontal line at p^*. See Figure 64.

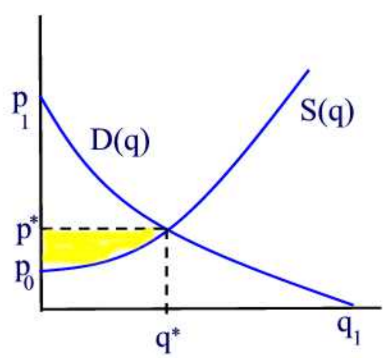


Figure 64

Example 34.1
The demand and supply equations are given by $D(q) = 60 - \frac{q^2}{10}$ and $S(q) = 30 + \frac{q^2}{5}$. Find the consumers' and producers' surplus at the equilibrium price.

Solution.
To find the Consumers and Producers surplus under equilibrium we first need to find the equilibrium point by setting supply=demand and solving for q:

$$30 + \frac{q^2}{5} = 60 - \frac{q^2}{10} \ \ \textit{implies} \ \ q^* = 10.$$

Substituting this into the supply (or demand) equation we find the equilibrium price $p^* = 50$. Now we use formulas of the Consumers and Producers surplus:

$$\begin{array}{rcccl} \textit{Consumers' Surplus} & : & \int_0^{10}\left[\left(60 - \frac{q^2}{10}\right) - 50\right] dq & = & 10q - \frac{q^3}{30}\Big|_0^{10} \approx 66.67 \\ \textit{Producers' Surplus} & : & \int_0^{10}\left[50 - \left(30 + \frac{q^2}{5}\right)\right] dq & = & 20q - \frac{q^3}{15}\Big|_0^{10} \approx 133.33 \ \blacksquare \end{array}$$

Recommended Problems (p. 264): 1, 3, 4, 5, 6.

35 Present and Future Value of a Continuous Income Stream

When an income stream flows into an investment, the investment grows because of the continuous flows of money and the interest compounded on the money invested. Thus, two functions are required: a function defining the flow of money, and a function defining a function multiplier.
In Section 7, we discussed the case of a discrete income stream. In this section, we cover the case of a continuous income stream. We will find the present value and the future value of a continuous income stream.
Let $S(t)$ be the flow rate in dollars per year. To find the present value of a continuous income stream over a period of M years we divide the interval $[0, M]$ into n equal subintervals each of length $\Delta t = \frac{M}{n}$ and with division points $0 = t_0 < t_1 < \cdots < t_n = M$. That is, over each time interval we are assuming a single payment is made. Assuming interest r is compounded continuously, the present value of the total money deposited is approximated by the following Riemann sum:

$$PV \approx S(t_1)e^{-rt_1}\Delta t + S(t_2)e^{-rt_2}\Delta t + \cdots S(t_n)e^{-rt_n}\Delta t = \sum_{i=1}^{n} S(t_i)e^{-rt_i}\Delta t.$$

Letting $\Delta t \to 0$, i.e. $n \to \infty$, we obtain

$$PV = \int_0^M S(t)e^{-rt}dt.$$

The future value is given by

$$FV = e^{rM}\int_0^M S(t)e^{-rt}dt.$$

Example 35.1
An investor is investing \$3.3 million a year in an account returning 9.4% APR. Assuming a continuous income stream and continuous compounding of interest, how much will these investments be worth 10 years from now?

Solution.
Using the formula for the future value defined above we find

$$FV = e^{.94}\int_0^{10} 3.3e^{-0.094t}dt \approx \$54.8 million. \blacksquare$$

Example 35.2
At what canstant, continuous rate must money be deposited into an account if the account contain \$20,000 in 5 years? The account earns 6% interest compounded continuously.

Solution.
Given $FV = \$20,000, M = 5, r = 0.06$. Since S is assumed to be constant then we have

$$20,000 = S\int_0^5 e^{-0.06t}dt.$$

Solving for S we find

$$S = \frac{20,000}{\int_0^5 e^{-0.06t}dt} \approx \$4,630 \textit{ per year.}\blacksquare$$

Recommended Problems (p. 267): 1, 2, 3, 4, 5, 7, 8, 9.

36 Constructing Antiderivatives Analytically

In this section we will find analytical expressions of antiderivatives. Recall that a function F is an **antiderivative** of a function f if $F'(x) = f(x)$. However, for any constant C, $F(x) + C$ is also an antiderivative of f. That is, there are infinitely many antiderivatives of a given function $f(x)$. They all differ by a constant and the family of antiderivatives is represented by $F(x) + C$. The notation of the general antiderivative is called an **indefinite integral** and is written

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is the symbol of integration, $f(x)$ is called the **integrand** and C is called the **constant of integration**. Keep in mind the relationship between $f(x)$ and $F(x)$ which is given by $F'(x) = f(x)$.

Warning: The indefinite integral is a short-hand notation for a family of functions $F(x) + C$ with the property $F'(x) = f(x)$ for all x. It is not to be confused with the definite integral $\int_a^b f(x)dx$ which is a real number.

Example 36.1
Show that $\int 0dx = C$.

Solution.
Since the derivative of a constant function is always zero then

$$\int 0dx = C.\blacksquare$$

Example 36.2
Show that $\int kdx = kx + C$ where k is a constant.

Solution.
Since the derivative of kx is just k then

$$\int kdx = kx + C.\blacksquare$$

Example 36.3
Show that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$.

Solution.
By the power rule, if $F(x) = \frac{x^{n+1}}{n+1}$ then $F'(x) = x^n$. Thus,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C. \blacksquare$$

Note that this formula is valid only if $n \neq -1$ for if $n = -1$ we would have $\frac{x^0}{0}$ which doesn't make sense. The case $n = -1$ is treated in the next problem.

Example 36.4
Show that

$$\int \frac{dx}{x} = \ln|x| + C.$$

Solution.
Suppose first that $x > 0$ so that $\ln|x| = \ln x$. Then $(\ln|x|)' = (\ln x)' = \frac{1}{x}$. Now, if $x < 0$ then $\ln|x| = \ln(-x)$ and by the chain rule $(\ln|x|)' = (\ln(-x))' = \frac{-1}{-x} = \frac{1}{x}$. Thus, in both cases $(\ln|x|)' = \frac{1}{x}$.$\blacksquare$

Example 36.5
Show that for $a \neq 0$, $\int e^{ax}dx = \frac{e^{ax}}{a} + C$.

Solution.
If a is a nonzero constant and $F(x) = \frac{e^{ax}}{a}$ then $F'(x) = e^{ax}$. This shows that

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C. \blacksquare$$

Example 36.6
Show that $\int \cos(ax)dx = \frac{\sin(ax)}{a} + C$ where $a \neq 0$.

Solution.
If a is a nonzero constant and $F(x) = \frac{\sin(ax)}{a}$ then $F'(x) = \cos(ax)$. This shows that

$$\int \cos(ax)dx = \frac{\sin(ax)}{a} + C. \blacksquare$$

Example 36.7
Show that $\int \sin(ax)dx = -\frac{\cos(ax)}{a} + C$ where $a \neq 0$.

Solution.
If a is a nonzero constant and $F(x) = -\frac{\cos(ax)}{a}$ then $F'(x) = \sin(ax)$. This shows that

$$\int \sin(ax)dx = -\frac{\cos(ax)}{a} + C. \blacksquare$$

Example 36.8
Show that $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$.

Solution.
Let $F(x) = \arcsin x$. Then $F'(x) = \frac{1}{\sqrt{1-x^2}}$. Thus,

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C. \blacksquare$$

Example 36.9
Show that $\int \frac{1}{1+x^2}dx = \arctan x + C$.

Solution.
Let $F(x) = \arctan x$. Then $F'(x) = \frac{1}{1+x^2}$. Thus,

$$\int \frac{1}{1+x^2}dx = \arctan x + C. \blacksquare$$

Properties of Indefinite Integrals

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx.$$

To see why this property is true, let $F(x)$ be an antiderivative of $f(x)$ and $G(x)$ be an antiderivative of $g(x)$. The result follows from the fact that $\frac{d}{dx}[F(x) \pm G(x)] = f(x) \pm g(x)$.

$$\int cf(x)dx = c\int f(x)dx.$$

To see this, suppose that $F(x)$ is an antiderivative of $f(x)$. Then $\int f(x)dx = F(x) + C$. But $\frac{d}{dx}(cF(x)) = cf(x)$ so that $cF(x)$ is an antiderivative of $cf(x)$, that is, $\int cf(x)dx = cF(x) + C'$. This implies

$$\int cf(x)dx = cF(x)+C' = c(\int f(x)dx - C)+C' = c\int f(x)dx - cC + C' = c\int f(x)dx.$$

Note that the constant $-cC + C'$ is ignored since a constant of integration will result from $\int f(x)dx$.

Example 36.10
Find
$$\int(\sin{(2x)} - e^{-3x} + \frac{3}{x} - \frac{5}{x^3})dx.$$

Solution.
Using the linearity property of indefinite integrals together with the formulas of integration obtained above we have

$$\begin{array}{rcl}\int(\sin{(2x)} - e^{-3x} + \frac{3}{x} - \frac{5}{x^3})dx & = & \int \sin{(2x)}dx - \int e^{-3x}dx + 3\int \frac{dx}{x} - 5\int x^{-3}dx \\ & = & -\frac{\cos{(2x)}}{2} + \frac{e^{-3x}}{3} + 3\ln{|x|} + \frac{5}{2x^2} + C\blacksquare\end{array}$$

Once we have found an antiderivative of $f(x)$, computing definite integrals is easy by the Fundamental Theorem of Calculus.

Example 36.11
Compute $\int_1^2 3x^2dx$.

Solution.
Since $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, then by FTC
$$\int_1^2 3x^2dx = x^3|_1^2 = 2^3 - 1^3 = 7.\blacksquare$$

Recommended Problems (pp. 281 - 2): 6, 8, 10, 13, 15, 22, 24, 32, 36, 38, 40,41, 48.

37 Integration by Substitution

The purpose of this section is to evaluate the integral

$$\int f'(g(x))g'(x)dx. \tag{4}$$

This is done, by letting $u = g(x)$. Then we define $du = g'(x)dx$. Hence, we have the following

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + C = f(g(x)) + C \tag{5}$$

The above procedure is referred to as the **method of integration by substitution**. Thus, when the integrand looks like a compound function times the derivative of the inside then try using substitution to integrate. Note also that this method of antidifferentiation reverses the chain rule of differentiation.
The following examples illustrate the use of this method.

Example 37.1
Find $\int 3x^2 \cos x^3 dx$.

Solution.
Let $u(x) = x^3$. Then $du = 3x^2dx$ and therefore

$$\int 3x^2 \cos x^3 dx = \int \cos u du = \sin u + C = \sin x^3 + C. \blacksquare$$

The method of substitution works even when the derivative of the inside is missing a constant factor as shown in the next example.

Example 37.2
Find $\int xe^{x^2+1}dx$.

Solution.
Letting $u(x) = x^2 + 1$ then $du = 2xdx$. Thus,

$$\begin{array}{rcl} \int xe^{x^2+1}dx &=& \frac{1}{2}\int 2xe^{x^2+1}dx \\ &=& \frac{1}{2}\int e^u du = \frac{e^u}{2} + C \\ &=& \frac{e^{x^2+1}}{2} + C \end{array}$$

You may wonder why $\frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$ and not $\frac{1}{2}\int e^u du = \frac{1}{2}(e^u + C) = \frac{e^u}{2} + \frac{C}{2}$. The convention is always to add C to whatever antiderivative we have calculated.$\blacksquare$

Example 37.3
Find $\int x^3\sqrt{x^4+5}dx.$

Solution.
Let $u = x^4+5$. Then $du = 4x^3dx$. Thus,

$$\begin{array}{rcl}\int x^3\sqrt{x^4+5}dx &=& \frac{1}{4}\int 4x^3\sqrt{x^4+5}dx\\ &=& \frac{1}{4}\int\sqrt{u}du = \frac{1}{4}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C\\ &=& \frac{1}{6}(x^4+5)^{\frac{3}{2}}+C \blacksquare\end{array}$$

Example 37.4
Find $\int \frac{e^x}{e^x+1}dx.$

Solution.
Let $u = e^x+1$. Then $du = e^xdx$. Thus,

$$\begin{array}{rcl}\int\frac{e^x}{e^x+1}dx &=& \int\frac{du}{u} = \ln|u|+C\\ &=& \ln|e^x+1|+C. \blacksquare\end{array}$$

Notice the pattern in the previous example: having a function in the denominator and its derivative in the numerator leads to a natural logarithm.

Example 37.5
Find $\int\sqrt{1+\sqrt{x}}dx.$

Solution.
Let $u = 1+\sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2(u-1)}$ or $dx = 2(u-1)du$. Thus,

$$\begin{array}{rcl}\int\sqrt{1+\sqrt{x}}dx &=& \int\sqrt{u}2(u-1)du = \int(2u\sqrt{u}-2\sqrt{u})du\\ &=& \int(2u^{\frac{3}{2}}-2u^{\frac{1}{2}})du\\ &=& 2\frac{u^{\frac{5}{2}}}{\frac{5}{2}}-2\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C\\ &=& \frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}}-\frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}}+C \blacksquare\end{array}$$

Recommended Problems (pp. 285 - 6): 1, 2, 3, 4, 6, 11, 13, 17, 19, 23, 26, 29, 31, 35.

38 Using the Fundamental Theorem to Find Definite Integrals

Recall the Fundamental Theorem of Calculus (abbreviated by FTC): If $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$. In particular, we have

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Once we have found an antiderivative of $f(x)$, computing definite integrals is easy by the Fundamental Theorem of Calculus.

Example 38.1
Compute $\int_1^2 3x^2 dx$.

Solution.
Since $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, then by FTC

$$\int_1^2 3x^2 dx = x^3|_1^2 = 2^3 - 1^3 = 7.\blacksquare$$

Example 38.2
Write a definite integral to represent the area under the graph of $f(t) = e^{0.5t}$ between $t = 0$ and $t = 4$. Use the Fundamental Theorem of Calculus to calculate the area.

Solution.
An antiderivative of $f(t)$ is $2e^{0.5t}$. Thus,

$$Area = \int_0^4 e^{0.5t}dt = 2e^{0.5t}\Big|_0^4 = 2e^2 - 2 \approx 12.778.\blacksquare$$

Next, we discuss the evaluation of a definite integral using the technique of substitution. From (5) we have that $f(g(x))$ is an antiderivative of the function $f'(g(x))g'(x)$. Applying the Fundamental Theorem of Calculus we can write

$$\int_a^b f'(g(x))g'(x)dx = f(g(x))|_a^b = f(g(b)) - f(g(a)).$$

If we let $u = g(x)$ then the previous formula reduces to

$$\int_a^b f'(g(x))g'(x)dx = f(g(b)) - f(g(a)) = \int_{g(a)}^{g(b)} f'(u)du.$$

Warning: When evaluating definite integrals, there is no constant of integration in the final answer.

Example 38.3
Compute $\int_0^2 xe^{x^2}dx$.

Solution.
Let $u(x) = x^2$. Then $du = 2xdx, u(0) = 0$, and $u(2) = 4$. Thus,

$$\int_0^2 xe^{x^2}dx = \frac{1}{2}\int_0^4 e^u du = \frac{e^u}{2}|_0^4 = \frac{1}{2}(e^4 - 1). \blacksquare$$

Example 38.4
Compute $\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{\cos^2 x}dx$.

Solution.
Let $u = \tan x$. Then $du = \frac{dx}{\cos^2 x}, u(0) = 0$, and $u(\frac{\pi}{4}) = 1$. Thus,

$$\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{\cos^2 x}dx = \int_0^1 u^3 du = \frac{u^4}{4}|_0^1 = \frac{1}{4}. \blacksquare$$

Recommended Problems (pp. 289 - 90): 1, 3, 5, 11, 13, 15, 17, 24, 25, 27, 28.

39 Finding Antiderivatives Graphically and Numerically

In this section we want to see how to reconstruct the graph of f given the graph of its derivative f'.

Example 39.1
The graph of $f'(x)$ is given in Figure 65.

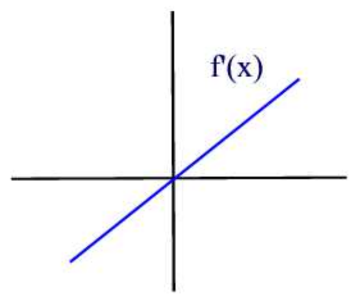


Figure 65

Sketch a graph of the function $f(x)$ satisfying $f(0) = 1$.

Solution.
Note that since $f'(x)$ is always increasing then $f''(x) > 0$ so that the graph of $f(x)$ is always concave up. Since $f'(x) < 0$ for $x < 0$ then $f(x)$ is decreasing there. Similarly, since $f'(x) > 0$ for $x > 0$ then $f(x)$ is increasing there. Since $f'(0) = 0$ and $f(x)$ is decreasing and then increasing we conclude that $x = 0$ is a minimum. A graph of $f(x)$ is given in Figure 66.■

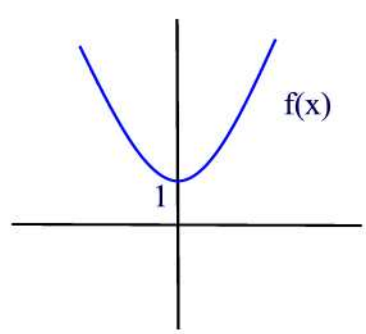


Figure 66

Example 39.2
The graph of $f'(x) = e^{-x^2}$ is given in Figure 67.

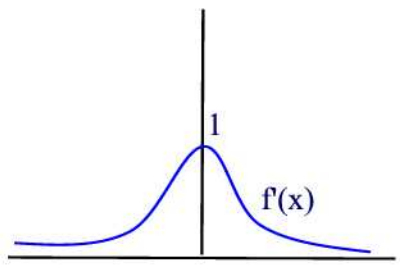


Figure 67

Sketch the graph of $f(x)$ satisfying $f(0) = 0$.

Solution.
Since $f'(x)$ is always positive then the graph of $f(x)$ is always increasing. Now, for $x < 0$, $f'(x)$ is increasing so that $f''(x) > 0$ and therefore $f(x)$ is concave up. For $x > 0$ the function $f'(x)$ is decreasing and so $f''(x) < 0$. That is, $f(x)$ is concave down there. Thus, $x = 0$ is a point of inflection. Finally, since $\lim_{x\to\pm\infty} f'(x) = 0$ then the graph of $f(x)$ levels off at both ends. See Figure 68.■

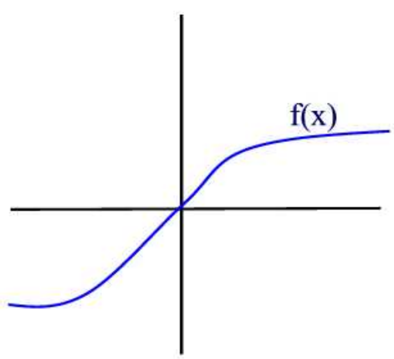


Figure 68

Next, we will reconstruct numerically the antiderivative f by using the Fundamental Theorem of Calculus: If $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$. In particular, we have

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Example 39.3
Suppose that $f'(t) = t\cos t$ and $f(0) = 2$. Find $f(0.3)$.

Solution.
Let $a = 0$ in the FTC to obtain

$$f(b) - f(0) = \int_0^b t\cos t dt.$$

But $f(0) = 2$ so the previous equation becomes

$$f(b) = 2 + \int_0^b t\cos t dt.$$

Thus,

$$f(0.3) = 2 + \int_0^{0.3} t\cos t dt.$$

Using the TI83 command $fnInt(x*cosx, x, 0, 0.3)$ we find that $\int_0^{0.3} t\cos t dt \approx 0.044$ so that $f(0.3) \approx 2.044.$■

Now, recall that for $f(x) \geq 0$ the definite integral $\int_a^b f(x)dx$ represents the area under the graph of $f(x)$ between the lines $x = a$ and $x = b$. If the region is below the x-axis then $\int_a^b f(x)dx$ is the negative of the area of that region.

Example 39.4
Figure 69 shows the graph of $f'(x)$. Suppose that $f(-1) = -2$. Find $f(0), f(1)$, and $f(3)$.

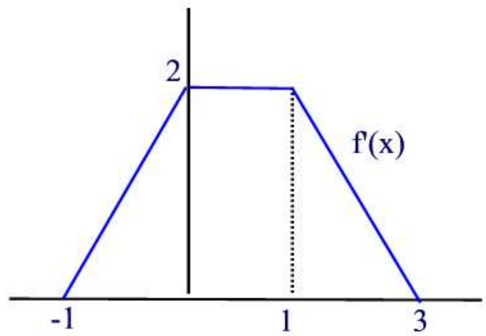


Figure 69

Solution.
By the FTC we have

$$f(b) = f(-1) + \int_{-1}^{b} f'(x)dx.$$

Thus,

$$\begin{array}{rcc} f(0) & = & f(-1) + \int_{-1}^{0} f'(x)dx \\ & = & -2 + \frac{1}{2}(1 \cdot 2) = -1 \\ f(1) & = & f(0) + \int_{0}^{1} f'(x)dx \\ & = & -1 + 1 \cdot 2 = 1 \\ f(3) & = & f(1) + \int_{1}^{3} f'(x)dx = 1 + \frac{1}{2}(2 \cdot 2) = 3 \end{array}$$

where we compute $\int_a^t f'(x)dx$ by determining the area between f' and the horizontal axis for $a \leq x \leq t$.■

Recommended Problems (pp. 294 - 6): 1, 3, 5, 9, 11, 17, 21, 22, 23, 24, 25.