Interest Formulas

\[ i = \frac{d}{1 - d}, \quad d = \frac{i}{i + 1} = 1 - v \]

\[ i - d = id, \text{ if compound interest} \]

\[ \frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} * \frac{d^{(m)}}{m} \]

Force of Interest

\[ \delta = \ln(1 + i), \quad (1 + i)^t = e^{\delta t} \]

\[ \left( 1 + \frac{i^{(m)}}{m} \right)^{mt} = 1 + i = \frac{1}{1 - d} = \left( 1 - \frac{d^{(n)}}{n} \right)^{-nt} = e^{\delta t} \]

\[ d < d^{(m)} < \delta < i^{(m)} < i, \quad m > 1 \]

For an accumulation function \( a(t) \),

\[ \delta_t = \frac{a'(t)}{a(t)} = \delta_t = -\frac{d}{dt}[a(t)]^{-1} \]

\[ a(t) = e^{\int_0^t \delta_t \, dr} \]

Date Conventions

Recall knuckle memory device. (February has 28/29 days)

- Exact
  - “actual/actual”
  - Uses exact days
  - 365 days in a nonleap year
  - 366 days in a leap year (divisible by 4)

- Ordinary
  - “30/360”
  - All months have 30 days
  - Every year has 360 days

- Banker’s Rule
  - “actual/360”
  - Uses exact days
  - Every year has 360 days

Basic Formulas

\[ \sum_{k=0}^{n} r^k = 1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \]

\[ \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \]

\[ \int a^x \, dx = \frac{a^x}{\ln(a)} + C \]
### Basic Equations

#### Immediate

- **Annuity Due**

  $a_{n|} = \frac{1 - (1 + i)^{-n}}{i}$  \hspace{1cm}  $\ddot{a}_{n|} = \frac{1 - (1 + i)^{-n}}{d}$

  $s_{n} = \frac{(1 + i)^n - 1}{i}$  \hspace{1cm}  $\ddot{s}_{n} = \frac{(1 + i)^n - 1}{d}$

  $a_{n} = (1 + i)^n a_{n|}$  \hspace{1cm}  $\ddot{a}_{n} = (1 + i)^n \ddot{a}_{n|}$

  $a_{m+n|} = a_{m|} + v^m a_{n|}$  \hspace{1cm}  $\ddot{a}_{m+n|} = \ddot{a}_{m|} + v^m \ddot{a}_{n|}$

  $\frac{1}{a_{n|}} = \frac{1}{s_{n}} + i$  \hspace{1cm}  $\frac{1}{\ddot{a}_{n|}} = \frac{1}{\ddot{s}_{n}} + d$

  $\ddot{a}_{n|} = (1 + i)a_{n|}$  \hspace{1cm}  $\ddot{s}_{n|} = (1 + i)s_{n|}$

  $\ddot{a}_{n} = 1 + a_{n-1|}$  \hspace{1cm}  $\ddot{s}_{n} = 1 + \ddot{s}_{n-1|}$

#### Due

- **Annuity Due**

  $a_{\infty|} = \frac{1}{i}$  \hspace{1cm}  $\ddot{a}_{\infty|} = \frac{1}{d}$

  $1 + a_{\infty} = \ddot{a}_{\infty}$

### Perpetuity

- **Perpetuity**

  $PV = \frac{1}{i}$  \hspace{1cm}  $\ddot{PV} = \frac{1}{d}$

### Annuities Payable More Frequently than Interest is Convertible

Let $m$ = the number of payments per interest conversion period

Hence the total number of annuity payments is $mn$

$$a_{n|}^{(m)} = PV \left( \text{payments of } \frac{1}{m} \text{ made at the end of each } m\text{th of an interest conversion period for } n \text{ periods} \right)$$

Coefficient of $a_{n|}^{(m)}$ is the total amount paid during one interest conversion period

### Annuities Payable Less Frequently than Interest is Convertible

Let $k$ = number of interest conversion periods in one payment period

Let $n$ = total number of conversion periods

Hence the total number of annuity payments is $\frac{n}{k}$

### Immediate

- **Annuity Due**

  $a_{n|}^{(m)} = \frac{1 - v^n}{i}$  \hspace{1cm}  $\ddot{a}_{n|}^{(m)} = \frac{1 - v^n}{d}$

  $s_{n}^{(m)} = \frac{(1 + i)^n - 1}{i}$  \hspace{1cm}  $\ddot{s}_{n}^{(m)} = \frac{(1 + i)^n - 1}{d}$

  $a_{n|}^{(m)} = \frac{i}{(m)} a_{n|}$  \hspace{1cm}  $\ddot{a}_{n|}^{(m)} = \frac{i}{(m)} \ddot{a}_{n|}$

  $s_{n|}^{(m)} = \frac{i}{(m)} s_{n|}$  \hspace{1cm}  $\ddot{s}_{n|}^{(m)} = \frac{i}{(m)} \ddot{s}_{n|}$

  $\frac{1}{a_{n|}^{(m)}} = \frac{1}{s_{n|}^{(m)}} + i$  \hspace{1cm}  $\frac{1}{\ddot{a}_{n|}^{(m)}} = \frac{1}{\ddot{s}_{n|}^{(m)}} + d$

### Perpetuity

- **Perpetuity**

  $a_{\infty|}^{(m)} = \frac{1}{i}$  \hspace{1cm}  $\ddot{a}_{\infty|}^{(m)} = \frac{1}{d}$
### Continuous Annuities

\[ \bar{a}_n = \frac{\delta}{\delta} a_n = \int_0^n e^{-\delta \int_0^t \delta_r dt} \]

### Varying Annuities

### Geometric

\[
\begin{align*}
    a &= 1 \\
    r &= 1 + k \quad k \neq i \\
    PV &= \frac{1 - \left(1 + \frac{k}{1+i}\right)^n}{i - k} \\
    \bar{p}V &= (1+i) \frac{1 - \left(1 + \frac{k}{1+i}\right)^n}{i - k} \\
    \text{If } k = i & \quad PV = nv \\
    \bar{p}V &= n \\
    a &= 1 \\
    r &= 1 - k \quad k \neq i \\
    PV &= \frac{1 - \left(1 - \frac{k}{1+i}\right)^n}{i + k} \\
    \bar{p}V &= (1+i) \frac{1 - \left(1 - \frac{k}{1+i}\right)^n}{i + k} \\
    \text{If } k = i & \quad PV = \frac{(1+i)^2}{2i} \left[ 1 - \left(1 + \frac{i}{1+i}\right)^n \right] \\
    \bar{p}V &= \frac{1}{2d} \left[ 1 - \left(1 + \frac{i}{1+i}\right)^n \right]
\end{align*}
\]

### Perpetuity

\[
\begin{align*}
    PV &= \frac{1}{i - k} \\
    \bar{p}V &= \frac{1}{i - k}
\end{align*}
\]

### Continuous Varying Annuities

Consider an annuity for \( n \) interest conversion periods in which payments are being made continuously at the rate \( f(t) \) and the interest rate is variable with force of interest \( \delta_t \).

\[
PV = \int_0^n f(t) e^{-\int_0^t \delta_r dt} dt
\]

Under compound interest, i.e. \( \delta_t = \ln (1 + i) \), the above becomes

\[
PV = \int_0^n f(t) v^t dt
\]
Rate of Return of an Investment

Yield rate, or IRR, is the interest rate at which

$PV(All\ Returns) = PV(All\ Contributions)$

Hence yield rates are solutions to $NPV(i)=0$

Discounted Cash Flow Technique

$$NPV(i) = \sum_{k=0}^{n} v^t k c_{tk}$$

Uniqueness of IRR

Theorem 1

- $\exists$ unique $i > -1$ s.t. $NPV(i) = 0$ if $c_t$ has only one sign change

Theorem 2

- Let $B_t$ be the outstanding balance at time $t$, i.e.
  - $B_t = c_0(1+i)^t + c_1(1+i)^{t-1} + \cdots + c_t$
  - $B_t > 0$ for $t < n$
- Then
  - $c_n = -B_{n-1}(1+i) < 0$
  - $i$ is unique

Interest Reinvested at a Different Rate

Invest 1 for $n$ periods at rate $i$, with interest reinvested at rate $j$

$$AV = 1 + is_{n|j}$$

Invest 1 at the end of each period for $n$ periods at rate $i$, with interest reinvested at rate $j$

$$AV = n + i(Is)_{n-1|j} = n + i \left[ \frac{s_{n|j} - n}{j} \right]$$

Invest 1 at the beginning of each period for $n$ periods at rate $i$, with interest reinvested at rate $j$

$$AV = n + i(Is)_{n|j} = n + i \left[ \frac{s_{n+i} - (n+1)}{j} \right]$$
Dollar-Weighted Interest Rate

A = the amount in the fund at the beginning of the period, i.e. t=0
B = the amount in the fund at the end of the period, i.e. t=1
I = the amount of interest earned during the period
c_t = the net amount of principal contributed at time t
C = Σc_t = total net amount of principal contributed during the period
i = the dollar-weighted rate of interest

Note: B = A+C+I

<table>
<thead>
<tr>
<th>Exact Equation</th>
<th>Simple Interest Approximation</th>
<th>Summation Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I = iA + \sum_t c_t[(1+i)^{1-t} - 1] )</td>
<td>( I \approx iA + \sum_t c_t(1-t)i )</td>
<td>The summation term is tedious.</td>
</tr>
<tr>
<td>( B = A(1+i) + \sum_t c_t (1 + i)^{1-t} )</td>
<td>( B \approx A(1+i) + \sum_t c_t [1 + (1-t)i] )</td>
<td>Define ( k = \frac{1}{c} \sum tc_t )</td>
</tr>
</tbody>
</table>

\[ i \approx \frac{I}{A + \sum c_t(1-t)} \]

\[ i \approx \frac{I}{A + (1-k)C} \]

\[ = \frac{kA + (1-k)B - (1-k)I}{A + B - I} \]

“Exposure associated with \( i \)” = \( A + \sum c_t(1-t) \)

If we assume uniform cash flow, then \( i \approx \frac{2I}{A + B - I} \)

Time-Weighted Interest Rate

Does not depend on the size or timing of cash flows.
Suppose \( n-1 \) transactions are made during a year at times \( t_1,t_2,\ldots,t_{n-1} \).
Let \( j_k = \) the yield rate over the \( k \)th subinterval
\( C_t = \) the net contribution at exact time \( t \)
\( B_t = \) the value of the fund before the contribution at time \( t \)
Then

\[ 1 + j_k = \frac{B_t}{B_{t-1} + C_{t-1}} \]

The overall yield rate \( i \) for the entire year is given by

\[ i = (1 + j_1)(1 + j_2) \ldots (1 + j_m) - 1 \]
Bonds

Notation
- \( P \) = the price paid for a bond
- \( F \) = the par value or face value
- \( C \) = the redemption value
- \( r \) = the coupon rate
- \( Fr \) = the amount of a coupon payment
- \( g \) = the modified coupon rate, defined by \( Fr/C \)
- \( i \) = the yield rate
- \( n \) = the number of coupons payment periods
- \( K \) = the present value, compute at the yield rate, of the redemption value at maturity, i.e. \( K=Cv^n \)
- \( G \) = the base amount of a bond, defined as \( G=Fr/i \). Thus, \( G \) is the amount which, if invested at the yield rate \( i \), would produce periodic interest payments equal to the coupons on the bond

Quoted yields associated with a bond
1) Nominal Yield
   a. Ratio of annualized coupon rate to par value
2) Current Yield
   a. Ratio of annualized coupon rate to original price of the bond
3) Yield to maturity
   a. Actual annualized yield rate, or IRR

Pricing Formulas
- Basic Formula
  - \( P = Fr\frac{a_{n|i}}{s_{k|i}} + Cv^n = Fr\frac{a_{n|i}}{s_{k|i}} + K \)
- Premium/Discount Formula
  - \( P = C + (Fr - Ci)\frac{a_{n|i}}{s_{k|i}} \)
- Base Amount Formula
  - \( P = G + (C - G)v^n \)
- Makeham Formula
  - \( P = K + \frac{g}{i} (C - K) \)

Yield rate and Coupon rate of Different Frequencies
Let \( n \) be the total number of yield rate conversion periods.
- Case 1: Each coupon period contains \( k \) yield rate periods
  - \( P = Fr\frac{a_{n|i}}{s_{k|i}} + Cv^n \)
- Case 2: Each yield period contains \( m \) coupon periods
  - \( P = Fr\frac{a_{n|i}(m)}{s_{k|i}} + Cv^n \)

Amortization of Premium or Discount
Let \( B_t \) be the book value after the \( t \)th coupon has just been paid, then
\[ B_t = Fr\frac{a_{n-t|i}}{s_{k|i}} + Cv^{n-t} \]
Let \( I_t \) denote the interest earned after the \( t \)th coupon has been made
\[ I_t = Cg + C(i - g)v^{n-t+1} \]
Let \( P_t \) denote the corresponding principal adjustment portion
\[ P_t = Fr - I_t = C(g - i)v^{n-t+1} \]

<table>
<thead>
<tr>
<th>Date</th>
<th>Coupon</th>
<th>Interest earned</th>
<th>Amount for Amortization of Premium</th>
<th>Book Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>June 1, 1996</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( B_0 = P )</td>
</tr>
<tr>
<td>Dec 1, 1996</td>
<td>Fr</td>
<td>( I_1 = iB_0 )</td>
<td>( P_1 = Fr - I_1 )</td>
<td>( B_1 = B_0 - P_1 )</td>
</tr>
<tr>
<td>June 1, 1997</td>
<td>Fr</td>
<td>( I_2 = iB_1 )</td>
<td>( P_2 = Fr - I_2 )</td>
<td>( B_2 = B_1 - P_2 )</td>
</tr>
</tbody>
</table>

Approximation Methods of Bonds’ Yield Rates

<table>
<thead>
<tr>
<th>Exact Formula</th>
<th>Approximation Method</th>
<th>Bond Salesman’s Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = g - \frac{k}{a_{n</td>
<td>i}} )</td>
<td>( i \approx g - \frac{n}{1 + \frac{n + 1}{2n}k} )</td>
</tr>
<tr>
<td>Where ( k = \frac{p-c}{c} )</td>
<td>Power series expansion</td>
<td>Equivalently ( i \approx \frac{nFr+C-P}{2n(P+C)} )</td>
</tr>
</tbody>
</table>
Valuation of Bonds between Coupon Payment Dates

- The purchase price for the bond is called the **flat price** and is denoted by $B^f_{t+k}$
- The price for the bond is the book value, or **market price**, and is denoted by $B^m_{t+k}$
- The part of the coupon the current holder would expect to receive as interest for the period is called the **accrued interest** or **accrued coupon** and is denoted by $Fr_k$

From the above definitions, it is clear that

$$B^f_{t+k} = B^m_{t+k} + Fr_k$$

**Theoretical Method**

The flat price should be the book value $B_t$ after the preceding coupon accumulated by $(1+i)^k$

- $B^f_{t+k} = (1+i)^kB_t$
- $Fr_k = Fr \left[ \frac{(1+i)^k-1}{i} \right]$
- $B^m_{t+k} = (1+i)^kB_t - Fr \left[ \frac{(1+i)^k-1}{i} \right]$

**Practical Method**

Uses the linear approximation $(1+i)^k \approx 1 + ki$

- $B^f_{t+k} = (1 + ki)B_t$
- $Fr_k = kFr$
- $B^m_{t+k} = (1 + ki)B_t - kFr = (1 - k)B_t + kB_{t+1}$

**Semi-theoretical Method**

Standard method of calculation by the securities industry. The flat price is determined as in the theoretical method, and the accrued coupon is determined as in the practical method.

- $B^f_{t+k} = (1+i)^kB_t$
- $Fr_k = kFr$
- $B^m_{t+k} = (1+i)^kB_t - kFr$

**Premium or Discount between Coupon Payment Dates**

$$Premium = B^m_{t+k} - C, \quad i < g$$
$$Discount = C - B^m_{t+k}, \quad i > g$$

**Callable Bonds**

The investor should assume that the issuer will redeem the bond to the disadvantage of the investor.

If the redemption value is the same at any call date, including the maturity date, then the following general principle will hold:

1) The call date will be at the earliest date possible if the bond was sold at a premium, which occurs when the yield rate is smaller than the coupon rate (issuer would like to stop repaying the premium via the coupon payments as soon as possible)
2) The call date will be at the latest date possible if the bond was sold at a discount, which occurs when the yield rate is larger than the coupon rate (issuer is in no rush to pay out the redemption value)

**Serial Bonds**

Serial bonds are bonds issued at the same time but with different maturity dates.

Consider an issue of serial bonds with $m$ different redemption dates. By Makeham’s formula,

$$P' = K' + \frac{g}{l}(C' - K')$$

where $P' = \sum_{t=1}^{m} P_t$ , $K' = \sum_{t=1}^{m} K_t$ , $C' = \sum_{t=1}^{m} C_t$
Loan Repayment Methods

Amortization Method

- Prospective Method
  - The outstanding loan balance at any time is equal to the present value at that time of the remaining payments.
- Retrospective Method
  - The outstanding loan balance at any time is equal to the original amount of the loan accumulated to that time less the accumulated value at that time of all payments previously made.

Consider a loan of $a_{\bar{n}|}$ at interest rate $i$ per period being repaid with payments of 1 at the end of each period for $n$ periods.

<table>
<thead>
<tr>
<th>Period</th>
<th>Payment amount</th>
<th>Interest paid</th>
<th>Principal repaid</th>
<th>Outstanding loan balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$a_{\bar{n}</td>
<td>}$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$ia_{\bar{n}</td>
<td>} = 1 - v^n$</td>
<td>$v^n$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$ia_{\bar{n-1}</td>
<td>} = 1 - v^{n-1}$</td>
<td>$v^{n-1}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>1</td>
<td>$ia_{\bar{n-k+1}</td>
<td>} = 1 - v^{n-k+1}$</td>
<td>$v^{n-k+1}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n-1$</td>
<td>1</td>
<td>$ia_{\bar{2}</td>
<td>} = 1 - v^2$</td>
<td>$v^2$</td>
</tr>
<tr>
<td>$n$</td>
<td>1</td>
<td>$ia_{\bar{1}</td>
<td>} = 1 - v$</td>
<td>$v$</td>
</tr>
<tr>
<td>Total</td>
<td>$n$</td>
<td>$n - a_{\bar{n}</td>
<td>}$</td>
<td>$a_{\bar{n}</td>
</tr>
</tbody>
</table>

Sinking Fund Method

Whereas with the amortization method the payment at the end of each period is $\frac{1}{a_{\bar{n}|}}$, in the sinking fund method, the borrower both deposits $\frac{1}{s_{\bar{n}|}}$ into the sinking fund and pays interest $i$ per period to the lender.

Example

Create a sinking fund schedule for a loan of $1000 repaid over four years with $i = 8\%$.

If $R$ is the sinking fund deposit, then $R = \frac{1000}{s_{\bar{4}|}} = $221.92

<table>
<thead>
<tr>
<th>Period</th>
<th>Interest paid</th>
<th>Sinking fund deposit</th>
<th>Interest earned on sinking fund</th>
<th>Amount in sinking fund</th>
<th>Net amount of loan</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
<td></td>
<td></td>
<td></td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>80</td>
<td>221.92</td>
<td>0</td>
<td>221.92</td>
<td>778.08</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>221.92</td>
<td>17.75</td>
<td>461.59</td>
<td>538.41</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
<td>221.92</td>
<td>36.93</td>
<td>720.44</td>
<td>279.56</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>221.92</td>
<td>57.64</td>
<td>1000</td>
<td>0</td>
</tr>
</tbody>
</table>
Measures of Interest Rate Sensitivity

**Stock**

- **Preferred Stock**
  - Provides a fixed rate of return
  - Price is the present value of future dividends of a perpetuity
  - \[ P = \frac{Fr}{i} \]

- **Common Stock**
  - Does not earn a fixed dividend rate
  - **Dividend Discount Model**
  - Value of a share is the present value of all future dividends
  - \[ P = D \lim_{n \to \infty} \frac{1-(1+i)^n}{1-k} = \frac{D}{i-k} \]

**Short Sales**

In order to find the yield rate on a short sale, we introduce the following notation:

- \( M = \) Margin deposit at t=0
- \( S_0 = \) Proceeds from short sale
- \( S_t = \) Cost to repurchase stock at time t
- \( d_t = \) Dividend at time t
- \( i = \) Periodic interest rate of margin account
- \( j = \) Periodic yield rate of short sale

\[ M(1+j)^n = M(1+i)^n + S_0 - S_n - \sum_{t=1}^n d_t (1+i)^{n-t} \]

**Inflation**

Given \( i' = \) real rate, \( i = \) nominal rate, \( r = \) inflation rate,

\[ 1 + i' = \frac{1 + i}{1 + r} \quad \text{or} \quad i' = \frac{i-r}{1+r} \]

**Fischer Equation**

A common approximation for the real interest rate:

\[ i' = i - r \]

**Duration**

- **Method of Equated Time (average term-to-maturity)**
  - \[ \bar{t} = \frac{\sum tR_t}{\sum R_t} \text{ where } R_1,R_2,\ldots,R_n \text{ are a series of payments made at times } 1,2,\ldots,n \]

- **Macaulay Duration**
  - \[ \bar{d} = \frac{\sum tv_tR_t}{\sum v_tR_t} = -(1+i) \frac{P'(i)}{P(i)} \text{, where } P(i) = \sum v_t R_t \]
  - \( \bar{d} \) is a decreasing function of \( i \)

- **Volatility (modified duration)**
  - \[ \bar{v} = -\frac{P'(i)}{P(i)} \]
  - \( \bar{v} = v \bar{d} \)
  - if \( P(i) \) is the current price of a bond, then
    - \[ \frac{P(i+\epsilon)-P(i)}{P(i)} = -\epsilon \bar{v} + \frac{\epsilon^2}{2} \bar{c} \]

- **Convexity**
  - \[ \bar{c} = \frac{P''(i)}{P(i)} \]

**Modified Duration and Convexity of a Portfolio**

Consider a portfolio consisting of \( n \) bonds. Let bond \( K \) have a current price \( P_k(i) \), modified duration \( \bar{v}_k(i) \), and convexity \( \bar{c}_k(i) \). Then the current value of the portfolio is

\[ P(i) = P_1(i) + P_2(i) + \cdots + P_n(i) \]

The modified duration \( \bar{v} \) of the portfolio is

\[ \bar{v} = -\sum \frac{P'_k(i)}{P(i)} \frac{P_k(i)}{P(i)} \bar{v}_1 + \cdots + \sum \frac{P_n(i)}{P(i)} \bar{v}_n \]

Similarly, the convexity \( \bar{c} \) of the portfolio is

\[ \bar{c} = -\sum \frac{P''_k(i)}{P(i)} \frac{P_k(i)}{P(i)} \bar{c}_1 + \cdots + \sum \frac{P'_n(i)}{P(i)} \bar{c}_n \]

Thus, the modified duration and convexity of a portfolio is the weighted average of the bonds’ modified durations and convexities respectively, using the market values of the bonds as weights.
Redington Immunization
Effective for small changes in interest rate $i$
Consider cash inflows $A_1,A_2,\ldots,A_n$ and cash outflows $L_1,L_2,\ldots,L_n$. Then the net cash flow at time $t$ is
$$R_t = A_t - L_t \text{ and } P(i) = \sum v^t R_t$$
Immunization conditions
- We need a local minimum at $i$
- $P(i) = 0$
  - The present value of cash inflows (assets) should be equal to the present value of cash outflows (liabilities)
- $P'(i) = 0$
  - The modified duration of the assets is equal to the modified duration of the liabilities
- $P''(i) > 0$
  - The convexity of $PV(Assets)$ should be greater than the convexity of $PV(Liabilities)$, i.e. asset growth > liability growth

Full Immunization
Effective for all changes in interest rate $i$
A portfolio is fully immunized if
$$PV_A(i + \varepsilon) > PV_L(i + \varepsilon) \ \forall \ \varepsilon > 0$$
Full immunization conditions for a single liability cash flow
1) $PV(Assets) = PV(Liabilities)$
2) $d_{Assets} = d_{Liabilities}$
3) Asset cash flow occurs before and after liability cash flow

Conditions (1) and (2) lead to the system
$$Ae^{a\delta} + Be^{-b\delta} = L_k$$
$$Aae^{a\delta} - Bbe^{-b\delta} = 0$$
where $\delta = \ln(1+i)$ and $k$=time of liability

Interest Yield Curves
The $k$-year forward $n$ years from now satisfied
$$(1 + i_n)^n(1 + i_{n+k})^k = (1 + i_{n+k})^{n+k}$$
where $i_n$ is the $t$-year spot rate

Dedication
Also called “absolute matching”
In this approach, a company structures an asset portfolio so that the cash inflow generated from assets will exactly match the cash outflow from liabilities.
**Option Styles**

- **European option** – Holder can exercise the option only on the expiration date
- **American option** – Holder can exercise the option anytime during the life of the option
- **Bermuda option** – Holder can exercise the option during certain pre-specified dates before or at the expiration date

<table>
<thead>
<tr>
<th>Buy</th>
<th>Write</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call</td>
<td>↑</td>
</tr>
<tr>
<td>Put</td>
<td>↓</td>
</tr>
</tbody>
</table>

**Floor** – own + buy put
**Cap** – short + buy call
**Covered Call** – stock + write call = write put
**Covered Put** – short + write put = write call

- **Cash-and-Carry** – buy asset + short forward contract
- **Synthetic Forward** – a combination of a long call and a short put with the same expiration date and strike price

\[ F_{0,T} = \text{no arbitrage forward price} \]

\[ \text{Call}(K, T) = \text{premium of call} \]

**Put-Call Parity**

\[ \text{Call}(K, T) - \text{Put}(K, T) = PV(F_{0,T} - K) \]

<table>
<thead>
<tr>
<th>Derivative Position</th>
<th>Maximum Loss</th>
<th>Maximum Gain</th>
<th>Position wrt Underlying Asset</th>
<th>Strategy</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Long Forward</strong></td>
<td>-Forward Price</td>
<td>Unlimited</td>
<td>Long(buy)</td>
<td>Guaranteed price</td>
<td>P_T-K</td>
</tr>
<tr>
<td><strong>Short Forward</strong></td>
<td>Unlimited</td>
<td>Forward Price</td>
<td>Short(sell)</td>
<td>Guaranteed price</td>
<td>K-P_T</td>
</tr>
<tr>
<td><strong>Long Call</strong></td>
<td>-FV(Premium)</td>
<td>Unlimited</td>
<td>Long(buy)</td>
<td>Insures against high price</td>
<td>max{0,P_T-K}</td>
</tr>
<tr>
<td><strong>Short Call</strong></td>
<td>Unlimited</td>
<td>FV(Premium)</td>
<td>Short(sell)</td>
<td>Sells insurance against high price</td>
<td>-min{0,P_T-K}</td>
</tr>
<tr>
<td><strong>Long Put</strong></td>
<td>-FV(Premium)</td>
<td>Strike Price – FV(Premium)</td>
<td>Short(sell)</td>
<td>Insures against low price</td>
<td>max{0,K-P_T}</td>
</tr>
<tr>
<td><strong>Short Put</strong></td>
<td>FV(Premium) – Strike Price</td>
<td>FV(Premium)</td>
<td>Long(buy)</td>
<td>Sells insurance against low price</td>
<td>-max{0,K-P_T}</td>
</tr>
</tbody>
</table>
(Buy index) + (Buy put option with strike K) = (Buy call option with strike K) + (Buy zero-coupon bond with par value K)
(Short index) + (Buy call option with strike K) = (Buy put option with strike K) + (Take loan with maturity of K)

Spread Strategy
Creating a position consisting of only calls or only puts, in which some options are purchased and some are sold
- Bull Spread
  - Investor speculates stock price will increase
    - Bull Call
      - Buy call with strike price \( K_1 \), sell call with strike price \( K_2 > K_1 \) and same expiration date
    - Bull Put
      - Buy put with strike price \( K_1 \), sell put with strike price \( K_2 > K_1 \) and same expiration date
  - Two profits are equivalent \( \rightarrow (\text{Buy } K_1 \text{ call}) + (\text{Sell } K_2 \text{ call}) = (\text{Buy } K_1 \text{ put}) + (\text{Sell } K_2 \text{ put}) \)
  - Profit function
    \[
    \begin{cases}
    -FV[\text{Call}(K_1, T) - \text{Call}(K_2, T)] & P_T \leq K_1 \\
    P_T - K_1 - FV[\text{Call}(K_1, T) - \text{Call}(K_2, T)] & K_1 < P_T < K_2 \\
    K_2 - K_1 - FV[\text{Call}(K_1, T) - \text{Call}(K_2, T)] & K_2 \leq P_T
    \end{cases}
    \]
- Bear Spread
  - Investor speculates stock price will decrease
  - Exact opposite of a bull spread
  - Bear Call
    - Sell \( K_1 \) call, buy \( K_2 \) call, where \( 0 < K_1 < K_2 \)
  - Bear Put
    - Sell \( K_1 \) put, buy \( K_2 \) put, where \( 1 < K_1 < K_2 \)

Long Box Spread

<table>
<thead>
<tr>
<th>Synthetic Long Forward</th>
<th>Synthetic Short Forward</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy call at ( K_1 )</td>
<td>Sell call at ( K_2 )</td>
</tr>
<tr>
<td>Sell put at ( K_1 )</td>
<td>Buy put at ( K_2 )</td>
</tr>
</tbody>
</table>

Regardless of spot price at expiration, the box spread guarantees a cash flow of \( K_2 - K_1 \) in the future.
Net premium of acquiring this position is \( \text{PV}(K_2 - K_1) \)
If \( K_1 < K_2 \), then lending money
Invest \( \text{PV}(K_2 - K_1) \), get \( K_2 - K_1 \)
If \( K_1 > K_2 \), then borrow money
Get \( \text{PV}(K_1 - K_2) \), pay \( K_1 - K_2 \)

Butterfly Spread
An insured written straddle
- Let \( K_1 < K_2 < K_3 \)
- Written straddle
  - Sell \( K_2 \) call, sell \( K_2 \) put
- Long strangle
  - Buy \( K_1 \) call, buy \( K_3 \) put
- Profit
  - Let \( FV[\text{Call}(K_1) - \text{Call}(K_2) - \text{Put}(K_2) + \text{Put}(K_3)] \)
    \[
    \begin{cases}
    K_3 - K_2 - FV & P_T \leq K_1 \\
    P_T - K_1 - K_2 + K_3 - FV & K_1 < P_T < K_2 \\
    K_2 - K_1 - FV & K_2 \leq P_T < K_3
    \end{cases}
    \]

Asymmetric Butterfly Spread
Define \( \lambda = \frac{K_3 - K_2}{K_3 - K_1} \), then \( K_2 = \lambda K_1 + (1 - \lambda) K_3 \)
For every written \( K_2 \) call, buy \( \lambda \) \( K_1 \) calls and buy \( (1 - \lambda) \) \( K_3 \) calls
Collar
Used to speculate on the decrease of the price of an asset
- Buy $K_1$-strike at-the-money put
- Sell $K_2$-strike out-of-the-money call
- $K_2 > K_1$
- $K_2 - K_1 = \text{collar width}$

Collared Stock
Collars can be used to insure assets we own
- Buy index
- Buy at-the-money $K_1$ put
- Buy out-of-the-money $K_2$ call
- $K_1 < K_2$

Zero-cost Collar
A collar with zero cost at time 0, i.e. zero net premium

Straddle
A bet on market volatility
- Buy $K$-strike call
- Buy $K$-strike put

Strangle
A straddle with lower premium cost
- Buy $K_1$-strike call
- Buy $K_2$ strike put
- $K_1 < K_2$

Profit Function
\[
\begin{align*}
\text{Collar} & : \begin{cases}
K_1 - P_T + FV[\text{Call}(K_2, T) - \text{Put}(K_1, T)] & P_T \leq K_1 \\
FV[\text{Call}(K_2, T) - \text{Put}(K_1, T)] & K_1 < P_T < K_2 \\
K_2 - P_T + FV[\text{Call}(K_2, T) - \text{Put}(K_1, T)] & K_2 \leq P_T
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{Collared Stock} & : \begin{cases}
K_1 + FV[\text{Call}(K_2, T) - \text{Put}(K_1, T) - P_0] & P_T \leq K_1 \\
P_T + FV[\text{Call}(K_2, T) - \text{Put}(K_1, T) - P_0] & K_1 < P_T < K_2 \\
K_2 + FV[\text{Call}(K_2, T) - \text{Put}(K_1, T) - P_0] & K_2 \leq P_T
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{Straddle} & : |P_T - K| - FV[\text{Call}(K, T) + \text{Put}(K, T)]
\end{align*}
\]

\[
\begin{align*}
\text{Strangle} & : \begin{cases}
K_1 - P_T - FV[\text{Call}(K_1, T) + \text{Put}(K_2, T)] & P_T \leq K_1 \\
-FV[\text{Call}(K_1, T) + \text{Put}(K_2, T)] & K_1 < P_T < K_2 \\
P_T - K_2 - FV[\text{Call}(K_1, T) + \text{Put}(K_2, T)] & K_2 \leq P_T
\end{cases}
\end{align*}
\]
Equity-linked CD (ELCD)

Payoff = \( P \left(1 + r \times \max \left( \frac{P_T}{P_0} - 1, 0 \right) \right) \)

Can financially engineer an equivalent by
- Buy zero-coupon bond at discount
- Use the difference to pay for an at-the-money call option

Prepaid Forward Contracts on Stock
- Let \( F^{P}_{0,T} \) denote the prepaid forward price for an asset bought at time 0 and delivered at time T
  - If no dividends, then \( F^{P}_{0,T} = S_0 \), otherwise arbitrage opportunities exist
  - If discrete dividends, then
    - \( F^{P}_{0,T} = S_0 - \sum_{i=1}^{n} PV(D_i) \)
  - If continuous dividends, then
    - Let \( \delta \) = yield rate, then the daily dollar dividend = \( \frac{\delta}{365} \) \( S_0 \) and 1 share at time 0 grows to \( e^{\delta T} \) shares at time T

Forward Contracts
- Discrete dividends
  - \( F_{0,T} = FV(F^{P}_{0,T}) = [S_0 - \sum_{i=1}^{n} D_i e^{-rT}] e^{rT} \)
- Continuous dividends
  - \( F_{0,T} = S_0 e^{(r-\delta)T} \)
- Forward premium = \( F_{0,T} / S_0 \)
- The annualized forward premium \( \alpha \) satisfies
  - \( F_{0,T} = S_0 e^{\alpha T} \) or \( \alpha = \frac{1}{T} \ln \left( \frac{F_{0,T}}{S_0} \right) \)
    - If no dividends, then \( \alpha = r \)
    - If continuous dividends, then \( \alpha = r - \delta \)

Financial Engineering of Synthetics
- (Forward) = (Stock) – (Zero-coupon bond)
  - Buy \( e^{-\delta T} \) shares of stock
  - Borrow \( S_0 e^{-\delta T} \) to pay for stock
  - Payoff = \( P_T - F_{0,T} \)
- (Stock) = (Forward) + (Zero-coupon bond)
  - Buy forward with price \( F_{0,T} = S_0 e^{(r-\delta)T} \)
  - Lend \( S_0 e^{-\delta T} \)
  - Payoff = \( P_T \)
- (Zero-coupon bond) = (Stock) – (Forward)
  - Buy \( e^{-\delta T} \) shares
  - Short one forward contract with price \( F_{0,T} \)
  - Payoff = \( F_{0,T} \)
  - If the rate of return on the synthetic bond is \( i \), then
    - \( S_0 e^{(i-\delta)T} = F_{0,T} \) or
    - Implied repo rate \( i = \frac{1}{T} \ln \left( \frac{F_{0,T}}{S_0} \right) \)