9 Solving Quasi-Linear First Order PDE via the Method of Characteristics

In this section we develop a method for finding the general solution of a quasi-linear first order partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{9.1}$$

This method is called the **method of characteristics** or **Lagrange's method.** This method consists of transforming the PDE to a system of ODEs which can be solved and the found solution is transformed into a solution for the original PDE.
The method of characteristics relies on a geometrical argument. A visualization of a solution is an integral surface with equation $z = u(x,y)$. An alternative representation of this integral surface is

$$F(x,y,z) = u(x,y) - z = 0.$$

That is, an integral surface is a level surface of the function $F(x,y,z)$.
Now, recall from vector calculus that the gradient vector to a level surface at the point (x,y,z) is a normal vector to the surface at that point. That is, the gradient is a vector normal to the tangent plane to the surface at the point (x,y,z). Thus, the normal vector to the surface $F(x,y,z) = 0$ is given by

$$\vec{n} = \nabla F = F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = u_x\vec{i} + u_y\vec{j} - \vec{k}.$$

Because of the negative $z-$ component, the vector $\vec{n}$ is pointing downward. Now, equation (9.1) can be written as the dot product

$$(a(x,y,u), b(x,y,u), c(x,y,u)) \cdot (u_x, u_y, -1) = 0$$

or

$$\vec{v} \cdot \vec{n} = 0$$

where $\vec{v} = a(x,y,u)\vec{i} + b(x,y,u)\vec{j} + c(x,y,u)\vec{k}$. Thus, $\vec{n}$ is normal to $\vec{v}$. Since $\vec{n}$ is normal to the surface $F(x,y,z) = 0$, the vector $\vec{v}$ must be tangential to the surface $F(x,y,z) = 0$ and hence must lie in the tangent plane to the surface at every point. Thus, to find a solution to (9.1) we need to find an integral surface such that the surface is tangent to the vector $\vec{v}$ at each of its point.

The required surface can be found as the union of integral curves, that is, curves that are tangent to $\vec{v}$ at every point on the curve. If an integral curve has a parametrization

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + u(t)\vec{k}$$

then the integral curve (i.e. the characteristic) is a solution to the ODE system

$$\frac{dx}{dt} = a(x,y,u), \ \frac{dy}{dt} = b(x,y,u), \ \frac{du}{dt} = c(x,y,u) \tag{9.2}$$

or in differential form

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}. \tag{9.3}$$

Equations (9.2) or (9.3) are called **characteristic equations**. Note that $u(t) = u(x(t), y(t))$ gives the values of u along a characteristic. Thus, along a characteristic, the PDE (9.1) degenerates to an ODE.

Example 9.1
Find the general solution of the PDE $yuu_x + xuu_y = xy$.

Solution.
The characteristic equations are $\frac{dx}{yu} = \frac{dy}{xu} = \frac{du}{xy}$. Using the first two fractions we find $x^2 - y^2 = k_1$. Using the last two fractions we find $u^2 - y^2 = f(x^2 - y^2)$. Hence, the general solution can be written as $u^2 = y^2 + f(x^2 - y^2)$, where f is an arbitrary differentiable single variable function ■

Example 9.2
Find the general solution of the PDE $x(y^2 - u^2)u_x - y(u^2 + x^2)y_y = (x^2 + y^2)u$.

Solution.
The characteristic equations are $\frac{dx}{x(y^2-u^2)} = \frac{dy}{-y(u^2+x^2)} = \frac{du}{(x^2+y^2)u}$. Using a property of proportions we can write

$$\frac{xdx + ydy + udu}{x^2(y^2-u^2) - y^2(u^2+x^2) + u^2(x^2+y^2)} = \frac{du}{(x^2+y^2)u}.$$

That is

$$\frac{xdx + ydy + udu}{0} = \frac{du}{(x^2+y^2)u}$$

or

$$xdx + ydy + udu = 0.$$

Hence, we find $x^2 + y^2 + u^2 = k_1$. Also,

$$\frac{\frac{dx}{x} - \frac{dy}{y}}{y^2 - u^2 + u^2 + x^2} = \frac{du}{(x^2 + y^2)u}$$

or

$$\frac{dx}{x} - \frac{dy}{y} = \frac{du}{u}.$$

Hence, we find $\frac{yu}{x} = k_2$. The general solution is given by

$$u(x, y) = \frac{x}{y} f(x^2 + y^2 + u^2)$$

where f is an arbitrary differentiable single variable function ■

Practice Problem

Problem 9.1
Find the general solution of the PDE $\ln(y+u)u_x + u_y = -1$.

Problem 9.2
Find the general solution of the PDE $x(y-u)u_x + y(u-x)u_y = u(x-y)$.

Problem 9.3
Find the general solution of the PDE $u(u^2+xy)(xu_x - yu_y) = x^4$.

Problem 9.4
Find the general solution of the PDE $(y+xu)u_x - (x+yu)u_y = x^2 - y^2$.

Problem 9.5
Find the general solution of the PDE $(y^2+u^2)u_x - xyu_y + xu = 0$.

Problem 9.6
Find the general solution of the PDE $u_t + uu_x = x$.

Problem 9.7
Find the general solution of the PDE $(y-u)u_x + (u-x)u_y = x - y$.

Problem 9.8
Solve

$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u.$$

Problem 9.9
Solve

$$\sqrt{1-x^2}u_x + u_y = 0.$$

Problem 9.10
Solve

$$u(x+y)u_x + u(x-y)u_y = x^2 + y^2.$$