8 Linear First Order PDE: The One Dimensional Spatial Transport Equations

Modeling is the process of writing a differential equation to describe a physical situation. In this section we derive the one-dimensional spatial transport equation and use the method of characteristics to solve it.

Linear Transport Equation for Fluid Flows

We shall describe the transport of a dissolved chemical by water that is traveling with uniform velocity c through a long thin tube G with uniform cross section A. (The very same discussion applies to the description of the transport of gas by air moving through a pipe.) We assume the velocity $c > 0$ is in the (rightward) positive direction of the $x-$axis. We will also assume that the concentration of the chemical is constant across the cross section A located at x so that the chemical changes only in the $x-$direction and thus the term one-dimensional spatial equation. This condition says that the quantity of the chemical in a portion of the tube is the same but it is traveling with time.

Let $u(x,t)$ be a continuously differentiable function denoting the concentration of the chemical (i.e. amount of chemical per unit volume) at position x and time t. Then at time t_0, the amount of chemical stored in a section of the tube between positions a and x_0 (see Figure 8.1) is given by the definite integral

$$\int_a^{x_0} Au(s,t_0)ds.$$

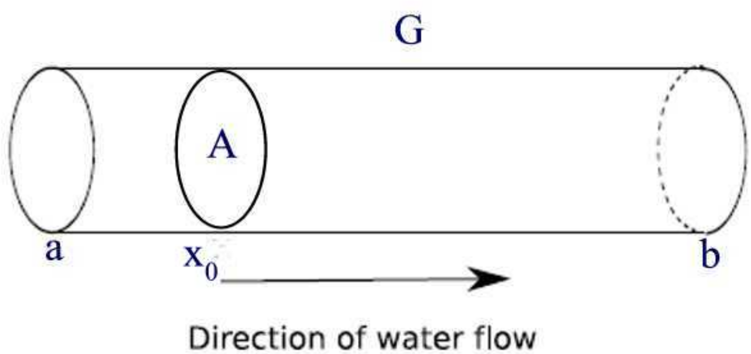

Figure 8.1

Since the water is flowing at a constant speed c, so at time $t_0 + h$ the same quantity of chemical will exist in the portion of the tube between $a + ch$ and $x_0 + ch$. That is,

$$\int_a^{x_0} Au(s, t_0)ds = \int_{a+ch}^{x_0+ch} Au(s, t_0 + h)ds.$$

Taking the derivative of both sides with respect to x_0 and using the Fundamental Theorem of Calculus, we find

$$u(x_0, t_0) = u(x_0 + ch, t_0 + h).$$

Now, taking the derivative of this last equation with respect to h and using the chain rule, with $x = x_0 + ch,\ t = t_0 + h$, we find

$$0 = u_t(x_0 + ch, t_0 + h) + cu_x(x_0 + ch, t_0 + h).$$

Taking the limit of this last equation as h approaches 0 and using the fact that u_t and u_x are continuous, we find

$$u_t(x_0, t_0) + cu_x(x_0, t_0) = 0. \tag{8.1}$$

Since x_0 and t_0 are arbitrary, Equation (8.1) is true for all (x, t). This equation is called the **transport equation** in one-dimensional space. It is a linear, homogeneous first order partial differential equation.
Note that (8.1) can be written in the form

$$< 1, c > \cdot < u_t, u_x >= 0$$

so that the left-hand side of (8.1) is the directional derivative of $u(t, x)$ at (t, x) in the direction of the vector $< 1, c >$.

Solvability via the method of characteristics
We will use the method of characteristics discussed in Chapter 7 to solve (8.1). The characteristic equations are

$$dt = \frac{dx}{c} = \frac{du}{0}.$$

Thus, to solve (8.1), we solve the system of ODEs

$$\frac{dt}{dx} = \frac{1}{c},\ \frac{du}{dx} = 0.$$

Solving the first equation, we find $x - ct = k_1$. Solving the second equation we find

$$u(x,t) = k_2 = f(k_1) = f(x - ct).$$

One can check that this is indeed a solution to (8.1). Indeed, by using the chain rule one finds

$$u_t = -cf'(x - ct) \quad \text{and} \quad u_x = f'(x - ct).$$

Hence, by substituting these results into the equation one finds

$$u_t + cu_x = -cf'(x - ct) + cf'(x - ct) = 0.$$

The solution $u(x,t) = f(x - ct)$ is called the **right traveling wave,** since the graph of the function $f(x - ct)$ at a given time t is the graph of $f(x)$ shifted to the right by the value ct. Thus, with growing time, the function $f(x)$ is moving without changes to the right at the speed c.

An initial value condition determines a unique solution to the transport equation as shown in the next example.

Example 8.1
Find the solution to $u_t - 3u_x = 0,\ u(x,0) = e^{-x^2}$.

Solution.
The characteristic equations lead to the ODEs

$$\frac{dt}{dx} = -\frac{1}{3},\ \frac{du}{dx} = 0.$$

Solving the first equation, we find $3t + x = k_1$. From the second equation, we find $u(x,t) = k_2 = f(k_1) = f(3t + x)$. From the initial condition, $u(x,0) = f(x) = e^{-x^2}$. Hence,

$$u(x,t) = e^{-(3t+x)^2} \blacksquare$$

Transport Equation with Decay
Recall from ODE that a function u is an exponential decay function if it satisfies the equation

$$\frac{du}{dt} = \lambda u,\ \lambda < 0.$$

A **transport equation with decay** is an equation given by

$$u_t + cu_x + \lambda u = f(x,t) \tag{8.2}$$

where $\lambda > 0$ and c are constants and f is a given function representing external resources. Note that the decay is characterized by the term λu.
Note that (8.2) is a first order linear partial differential equation that can be solved by the method of characteristics by solving the chracteristic equations

$$\frac{dx}{c} = \frac{dt}{1} = \frac{du}{f(x,t) - \lambda u}.$$

Example 8.2
Find the general solution of the transport equation

$$u_t + u_x + u = t.$$

Solution.
The characteristic equations are

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{t-u}.$$

From the equation $dx = dt$ we find $x - t = k_1$. Using a property of proportions we can write

$$\frac{dt}{1} = \frac{du}{t-u} = \frac{dt - du}{1 - t + u} = -\frac{d(1-t+u)}{1-t+u}.$$

Thus, $1 - t + u = k_2 e^{-t} = f(x-t)e^{-t}$ or $u(x,t) = t - 1 + f(x-t)e^{-t}$ where f is a differentiable function of one variable ■

Practice Problems

Problem 8.1
Find the solution to $u_t + 3u_x = 0,\ u(x, 0) = \sin x$.

Problem 8.2
Solve the equation $au_x + bu_y + cu = 0$.

Problem 8.3
Solve the equation $u_x+2u_y = \cos(y - 2x)$ with the initial condition $u(0, y) = f(y)$, where $f : \mathbb{R} \to \mathbb{R}$ is a given function.

Problem 8.4
Show that the initial value problem $u_t + u_x = x,\ u(x, x) = 1$ has no solution.

Problem 8.5
Solve the transport equation $u_t + 2u_x = -3u$ with initial condition $u(x, 0) = \frac{1}{1+x^2}$.

Problem 8.6
Solve $u_t + u_x - 3u = t$ with initial condition $u(x, 0) = x^2$.

Problem 8.7
Show that the decay term λu in the transport equation with decay

$$u_t + cu_x + \lambda u = 0$$

can be eliminated by the substitution $w = ue^{\lambda t}$.

Problem 8.8 *(Well-Posed)*
Let u be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x)$$

and v be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = g(x)$$

where f and g are continuously differentiable functions.
(a) Show that $w(x,t) = u(x,t) - v(x,t)$ is the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x,0) = f(x) - g(x)$$

(b) Write an explicit formula for w in terms of f and g.
(c) Use (b) to conclude that the transport problem is well-posed. That is, a small change in the initial data leads to a small change in the solution.

Problem 8.9
Solve the initial boundary value problem

$$u_t + cu_x = -\lambda u, \ \ x > 0, \ \ t > 0$$

$$u(x,0) = 0, \ \ u(0,t) = g(t), \ \ t > 0.$$

Problem 8.10
Solve the first-order equation $2u_t + 3u_x = 0$ with the initial condition $u(x,0) = \sin x$.

Problem 8.11
Solve the PDE $u_x + u_y = 1$.