7 Solvability of Semi-linear First Order PDEs

In this section we discuss the solvability of the semi-linear first order PDE

$$a(x,y)u_x + b(x,y)u_y = f(x,y,u) \tag{7.1}$$

via the **method of characteristics**.
To solve (7.1), we proceed as follows. Suppose we have found a solution $u(x,y)$ to (7.1). This solution may be interpreted geometrically as a surface in (x,y,z) space called the **integral surface** where $z = u(x,y)$. This integral surface can be viewed as the level surface of the function

$$F(x,y,z) = u(x,y) - z = 0.$$

Then equation (7.1) can be written as the dot product

$$\vec{v} \cdot \vec{n} = 0 \tag{7.2}$$

where $\vec{v} =< a, b, f >$ is the **characteristic direction** and $\vec{n} = \nabla F(x,y,z) =< u_x, u_y, -1 >$. Note that $\vec{n}$ is normal to the surface $F(x,y,z) = 0$. Hence, $\vec{n}$ is normal to $\vec{v}$ and this implies that $\vec{v}$ is tangent to the surface $z = u(x,y)$ at (x,y,z). So our task to finding a solution to (7.1) is equivalent to finding a surface $\mathcal{S}$ such that at every point on the surface the vector

$$\vec{v} = a\vec{i} + b\vec{j} + f(x,y,u)\vec{k}.$$

is tangent to the surface. How do we construct such a surface? The idea is to find the integral curves of the vector field $\vec{v}$ (see Section 6.2) and then patch all these curves together to obtain the desired surface.
To this end, we start first by constructing a curve Γ parametrized by t such that at each point of Γ the vector $\vec{v}$ is tangent to Γ. A parametrization of this curve is given by the vector function

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + u(t)\vec{k}.$$

Then the tangent vector is

$$\vec{r'}(t) = \frac{d}{dt}(\vec{r}(t)) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{du}{dt}\vec{k}.$$

Hence, the vectors $\vec{r'}(t)$ and $\vec{v}$ are parallel so these two vectors are proportional and this leads to the ODE system

$$\frac{\frac{dx}{dt}}{a} = \frac{\frac{dy}{dt}}{b} = \frac{\frac{du}{dt}}{f(x,y,u)} \tag{7.3}$$

or in differential form

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y,u)}. \tag{7.4}$$

By solving the system (7.3) or (7.4), we are assured that the vector $\vec{v}$ is tangent to the curve Γ which in turn lies in the solution surface $\mathcal{S}$. In our context, integral curves are called **characteristic curves** or simply **characteristics** of the PDE (7.1). We call (7.3) the **characteristic equations**. The projection of Γ into the $xy-$plane is called the **projected characteristic curve**.

Once we have found the characteristic curves, the surface $\mathcal{S}$ is the union of these characteristic curves. In summary, by introducing these characteristic equations, we have reduced our partial differential equation to a system of ordinary differential equations. We can use ODE theory to solve the characteristic equations, then piece together these characteristic curves to form a surface. Such a surface will provide us with a solution to our PDE.

Remark 7.1

Solving $\frac{dy}{dx} = \frac{b}{a}$ one obtains the general solution $h(x,y) = k_1$ where k_1 is constant. Likewise, solving $\frac{du}{dx} = \frac{f}{a}$ one obtains the general solution $j(x,y,u) = k_2$ where k_2 is a constant. The constant k_2 is a function of k_1. For the sake of discussion, suppose that $h(x,y) = k_1$ can be expressed as $y = g(x,k_1)$. Then, the y in $\frac{du}{dx} = \frac{f}{a}$ is being replaced by $g(x,k_1)$ so that the constant in $j(x,y,u) = k_2$ will depend on k_1.

Example 7.1

Find the general solution to $au_x + bu_y = 0$ where a and b are constants with $a \neq 0$.

Solution.

From (7.3) we can write $\frac{dy}{dx} = \frac{b}{a}$ which yields $bx - ay = k_1$ for some arbitrary constant k_1. From $\frac{du}{dx} = 0$ we find $u(x,y) = k_2$ where k_2 is a constant. That is, $u(x,y)$ is constant on Γ. Since $(0, -\frac{k_1}{a}, k_2)$ is on Γ, we have

$$u(x,y) = u(0, -\frac{k_1}{a}) = k_2$$

which shows that k_2 is a function of k_1. Hence,

$$u(x,y) = f(k_1) = f(bx - ay)$$

where f is a differentiable function in one variable ■

In the next example, we show how the initial value problem for the PDE determines the function f.

Example 7.2
Find the unique solution to $au_x + bu_y = 0$, where a and b are constants with $a \neq 0$, with the initial condition $u(x,0) = g(x)$.

Solution.
From the previous example, we found $u(x,y) = f(bx - ay)$ for some differentiable function f. Since $u(x,0) = g(x)$, we find $g(x) = f(bx)$ or $f(x) = g\left(\frac{x}{b}\right)$ assuming that $b \neq 0$. Thus,

$$u(x,y) = g\left(x - \frac{a}{b}y\right) \blacksquare$$

Example 7.3
Find the solution to $-3u_x + u_y = 0,\ u(x,0) = e^{-x^2}$.

Solution.
We have $a = -3, b = 1$ and $g(x) = e^{-x^2}$. The unique solution is given by

$$u(x,y) = e^{-(x+3y)^2} \blacksquare$$

Example 7.4
Find the general solution of the equation

$$xu_x + yu_y = xe^{-u},\ x > 0.$$

Solution.
We have $a(x,y) = x,\ b(x,y) = y$, and $f(x,y,u) = xe^{-u}$. So we have to solve the system

$$\frac{dy}{dx} = \frac{y}{x},\ \frac{du}{dx} = e^{-u}.$$

From the first equation, we can use the separation of variables method to find $y = k_1 x$ for some constant k_1. Solving the second equation by the method of separation of variables, we find

$$e^u - x = k_2.$$

But $k_2 = g(k_1)$ so that

$$e^u - x = g(k_1) = g\left(\frac{y}{x}\right)$$

where g is a differentiable function of one variable ■

Example 7.5
Find the general solution of the equation

$$u_x + u_y - u = y.$$

Solution.
The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u+y} = \frac{d(u+y+1)}{u+y+1}.$$

Solving the equation $\frac{dy}{dx} = 1$ we find $y - x = k_1$. Solving the equation $dx = \frac{d(u+y+1)}{u+y+1}$, we find $u + y + 1 = k_2 e^x = f(y-x)e^x$, where f is a differentiable function of one variable. Hence,

$$u = -(1+y) + f(y-x)e^x \ ■$$

Example 7.6
Find the general solution to $x^2 u_x + y^2 u_y = (x+y)u$.

Solution.
Using properties of proportions[1] we have

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u} = \frac{dx - dy}{x^2 - y^2}.$$

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[1] If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a \pm b}{b} = \frac{c \pm d}{d}$. Also, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{\alpha a + \beta c + \gamma e}{\alpha b + \beta d + \gamma f}$.

Solving $\frac{dy}{dx} = \frac{y^2}{x^2}$ by the method of separation of variables we find $\frac{1}{x} - \frac{1}{y} = k_1$. From the equation $\frac{du}{(x+y)u} = \frac{d(x-y)}{x^2-y^2}$ we find

$$\frac{du}{u} = \frac{d(x-y)}{x-y}$$

which implies

$$u = k_2(x-y) = f\left(\frac{1}{x} - \frac{1}{y}\right)(x-y) \blacksquare$$

Example 7.7
Find the solution satisfying $yu_x + xu_y = x^2 + y^2$ subject to the conditions $u(x,0) = 1 + x^2$ and $u(0,y) = 1 + y^2$.

Solution.
Solving the equation $\frac{dy}{dx} = \frac{x}{y}$ we find $x^2 - y^2 = k_1$. On the other hand, we have

$$\begin{aligned} du =& y^{-1}(x^2+y^2)dx \\ =& ydx + x^2y^{-1}dx \\ =& ydx + x^2y^{-1}\left(\frac{y}{x}dy\right) \\ =& ydx + xdy = d(xy). \end{aligned}$$

Hence,

$$u(x,y) = xy + f(x^2 - y^2).$$

From $u(x,0) = 1 + x^2$ we find $f(x) = 1 + x,\ x \geq 0$. From $u(0,y) = 1 + y^2$ we find $f(y) = 1 - y,\ y \leq 0$. Hence, $f(x) = 1 + |x|$ and

$$u(x,y) = xy + |x^2 - y^2| \blacksquare$$

Remark 7.2
The method of characteristics discussed in this section applies as well to any quasi-linear first order PDE. See Chapter 9.

Practice Problems

Problem 7.1
Solve $u_x + yu_y = y^2$ with the initial condition $u(0, y) = \sin y$.

Problem 7.2
Solve $u_x + yu_y = u^2$ with the initial condition $u(0, y) = \sin y$.

Problem 7.3
Find the general solution of $yu_x - xu_y = 2xyu$.

Problem 7.4
Find the integral surface of the IVP: $xu_x + yu_y = u,\ u(x, 1) = 2 + e^{-|x|}$.

Problem 7.5
Find the unique solution to $4u_x + u_y = u^2,\ u(x, 0) = \frac{1}{1+x^2}$.

Problem 7.6
Find the unique solution to $e^{2y}u_x + xu_y = xu^2,\ u(x, 0) = e^{x^2}$.

Problem 7.7
Find the unique solution to $xu_x + u_y = 3x - u,\ u(x, 0) = \tan^{-1} x$.

Problem 7.8
Solve: $xu_x - yu_y = 0,\ u(x, x) = x^4$.

Problem 7.9
Find the general solution of $yu_x - 3x^2yu_y = 3x^2u$.

Problem 7.10
Find $u(x, y)$ that satisfies $yu_x + xu_y = 4xy^3$ subject to the boundary conditions $u(x, 0) = -x^4$ and $u(0, y) = 0$.