6 A Review of Multivariable Calculus

In this section, we recall some concepts from vector calculus that we encounter later in the book.

6.1 Multiplication of Vectors: The Scalar or Dot Product

Is there such thing as multiplying a vector by another vector? The answer is yes. As a matter of fact there are two types of vector multiplication. The first one is known as **scalar** or **dot product**[1] and produces a scalar; the second is known as the **vector** or **cross product** and produces a vector. In this section we will discuss the former one leaving the latter one for the next section.
One of the motivation for using the dot product is the physical situation to which it applies, namely that of computing the work done on an object by a given force over a given distance, as shown in Figure 6.1.1.

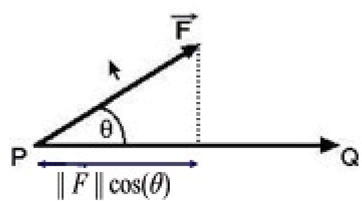


Figure 6.1.1

Indeed, the work W is given by the expression

$$W = ||\vec{F}||\ ||\overrightarrow{PQ}|| \cos\theta$$

where $||\vec{F}|| \cos\theta$ is the component of $\vec{F}$ in the direction of $\overrightarrow{PQ}$.
Thus, we define the **dot product** of two vectors $\vec{u}$ and $\vec{v}$ to be the number

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta, \quad 0 \leq \theta \leq \pi$$

[1] Also called **inner product.**

where θ is the angle between the two vectors as shown in Figure 6.1.2.

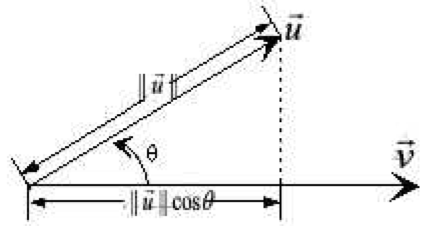


Figure 6.1.2

The above definition is the geometric definition of the dot product. We next provide an algebraic way for computing the dot product. Indeed, let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Then $\vec{v} - \vec{u} = (v_1 - u_1)\vec{i} + (v_2 - u_2)\vec{j} + (v_3 - u_3)\vec{k}$. Moreover, we have $||\vec{u}||^2 = u_1^2 + u_2^2 + u_3^2$, $||\vec{v}||^2 = v_1^2 + v_2^2 + v_3^2$ and

$$\begin{aligned} ||\vec{v} - \vec{u}||^2 =& (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 \\ =& v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2. \end{aligned}$$

Now, applying the Law of Cosines to Figure 6.1.3 we can write

$$||\vec{v} - \vec{u}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

Thus, by substitution we obtain

$$v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2 = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2||\vec{u}||\ ||\vec{v}|| \cos\theta$$

or

$$||\vec{u}||\ ||\vec{v}|| \cos\theta = u_1v_1 + u_2v_2 + u_3v_3$$

so that we can define the dot product algebraically by

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

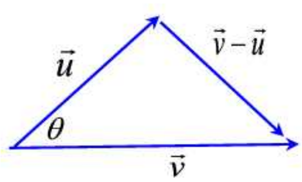


Figure 6.1.3

Example 6.1.1
Compute the dot product of $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}$ and $\vec{v} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}$ and the angle between these vectors.

Solution.
We have

$$\vec{u} \cdot \vec{v} = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

We also have

$$||\vec{u}||^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{2}$$
$$||\vec{v}||^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1 = \frac{3}{2}.$$

Thus,

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \; ||\vec{v}||} = \frac{2\sqrt{2}}{3}.$$

Hence,

$$\theta = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) \approx 0.34 \; rad \approx 19.5^\circ \; \blacksquare$$

Remark 6.1.1
The algebraic definition of the dot product extends to vectors with any number of components.

Next, we discuss few properties of the dot product.

Theorem 6.1.1
For any vectors $\vec{u}, \vec{v}$, and $\vec{w}$ and any scalar λ we have
(i) Commutative law: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
(ii) Distributive law: $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
(iii) $\vec{u} \cdot (\lambda\vec{v}) = (\lambda\vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v})$.
(iv) Magnitude: $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$.
(v) Two nonzero vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** or **perpendicular** if and only if $\vec{u} \cdot \vec{v} = 0$.
(vi)) Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \cdot \vec{v} = \pm||\vec{u}|| \; ||\vec{v}||$.
(vii) $\vec{0} \cdot \vec{v} = \vec{0}$.

Proof.
Write $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$. Then
(i) $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_2v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \vec{v} \cdot \vec{u}$ since product of numbers is commutative.
(ii) $(\vec{u} + \vec{v}) \cdot \vec{w} = ((u_1 + v_1)\vec{i} + (u_2 + v_2)\vec{j} + (u_2 + v_3)\vec{k}) \cdot (w_1\vec{i} + w_2\vec{j} + w_3\vec{k}) = (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + (u_3 + v_3)w_3 = u_1w_1 + u_2w_2 + u_3w_3 + v_1w_1 + v_2w_2 + v_3w_3 = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
(iii) $\vec{u} \cdot (\lambda\vec{v}) = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \cdot (\lambda v_1\vec{i} + \lambda v_2\vec{j} + \lambda v_3\vec{k}) = \lambda u_1v_1 + \lambda u_2v_2 + \lambda u_3v_3 = \lambda(u_1v_1 + u_2v_2 + u_3v_3) = \lambda(\vec{u} \cdot \vec{v})$.
(iv) $||\vec{u}||^2 = \vec{u} \cdot \vec{u} \cos 0 = \vec{u} \cdot \vec{u}$.
(v) If $\vec{u}$ and $\vec{v}$ are perpendicular then the cosine of their angle is zero and so the dot product is zero. Conversely, if the dot product of the two vectors is zero then the cosine of their angle is zero and this happens only when the two vectors are perpendicular.
(vi) If $\vec{u}$ and $\vec{v}$ are parallel then the cosine of their angle is either 1 or -1. That is, $\vec{u} \cdot \vec{v} = \pm||\vec{u}||\ ||\vec{v}||$. Conversely, if $\vec{u} \cdot \vec{v} = \pm||\vec{u}||\ ||\vec{v}||$ then $\cos\theta = \pm 1$ and this implies that either $\theta = 0$ or $\theta = \pi$. In either case, the two vectors are parallel.
(vii) In 3-D, $\vec{0} =< 0, 0, 0 >$ and $\vec{v} =< a, b, c >$ so that $\vec{0} \cdot \vec{v} = (0 \times a)\vec{i} + (0 \times b)\vec{j} + (0 \times c)\vec{k} = \vec{0}$ ■

Remark 6.1.2
Note that the unit vectors $\vec{i}, \vec{j}, \vec{k}$ associated with the coordinate axes satisfy the equalities

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \text{ and } \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{i} \cdot \vec{k} = 0.$$

Example 6.1.2
(a) Show that the vectors $\vec{u} = 3\vec{i} - 2\vec{j}$ and $\vec{v} = 2\vec{i} + 3\vec{j}$ are perpendicular.
(b) Show that the vectors $\vec{u} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ and $\vec{v} = -3\vec{i} - 9\vec{j} + 6\vec{k}$ are parallel.

Solution.
(a) We have: $\vec{u} \cdot \vec{v} = 3(2) - 2(3) = 0$. Hence $\vec{u}$ is perpendicular to $\vec{v}$.
(b) We have:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||} = \frac{2(-3) + (6)(-9) - 4(6)}{[\sqrt{2^2 + (6)^2 + (-4)^2}][\sqrt{(-3)^2 + (-9)^2 + 6^2}]} = -1.$$

Hence, $\theta = \pi$ so that the two vectors are parallel. Another way to see that the vectors are parallel is to notice that $\vec{u} = -\frac{2}{3}\vec{v}$ ■

Projection of a vector onto a line

The **orthogonal projection** of a vector along a line is obtained by taking a vector with same length and direction as the given vector but with its tail on the line and then dropping a perpendicular onto the line from the tip of the vector. The resulting vector on the line is the vector's orthogonal projection or simply its projection. See Figure 6.1.4.

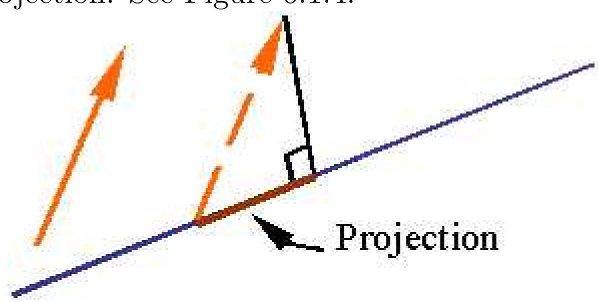


Figure 6.1.4

Now, if $\vec{u}$ is a unit vector along the line of projection and if $\vec{v}_{parallel}$ is the vector projection of $\vec{v}$ onto $\vec{u}$ then

$$\vec{v}_{parallel} = (||\vec{v}|| \cos\theta)\vec{u} = (\vec{v} \cdot \vec{u})\vec{u}.$$

See Figure 6.1.5. Also, the component perpendicular to $\vec{u}$ is given by

$$\vec{v}_{perpendicular} = \vec{v} - \vec{v}_{parallel}.$$

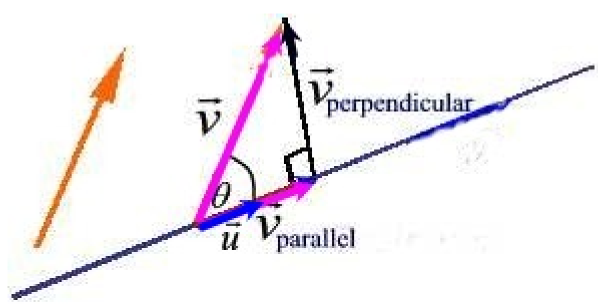


Figure 6.1.5

We call $\text{Comp}_{\vec{u}}\vec{v} = \vec{v}\cdot\vec{u}$ the the **scalar projection** of $\vec{v}$ onto $\vec{u}$. We call the vector $\text{Proj}_{\vec{u}}\vec{v} = \vec{v}_{parallel}$ the **vector projection** of $\vec{v}$ onto $\vec{u}$.
It follows that the vector $\vec{v}$ can be written in terms of $\vec{v}_{parallel}$ and $\vec{v}_{perpendicular}$

$$\vec{v} = \vec{v}_{parallel} + \vec{v}_{perpendicular}.$$

Example 6.1.3
Write the vector $\vec{v} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular to $\vec{w} = 2\vec{i} - 4\vec{j} + \vec{k}$.

Solution.
Let $\vec{u} = \frac{\vec{w}}{||\vec{w}||} = \frac{2}{\sqrt{21}}\vec{i} - \frac{4}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}$. Then,

$$\vec{v}_{parallel} = (\vec{v}\cdot\vec{u})\vec{u} = \left(\frac{6}{\sqrt{21}} - \frac{8}{\sqrt{21}} - \frac{6}{\sqrt{21}}\right)\vec{u} = -\frac{16}{21}\vec{i} + \frac{32}{21}\vec{j} - \frac{8}{21}\vec{k}.$$

Also,

$$\begin{aligned}\vec{v}_{perpendicular} =& \vec{v} - \vec{v}_{parallel} = \left(3 + \frac{16}{21}\right)\vec{i} + \left(2 - \frac{32}{21}\right)\vec{j} + \left(-6 + \frac{8}{21}\right)\vec{k}\\ =& \frac{79}{21}\vec{i} + \frac{10}{21}\vec{j} - \frac{118}{21}\vec{k}.\end{aligned}$$

Hence,

$$\vec{v} = \vec{v}_{parallel} + \vec{v}_{perpendicular} \blacksquare$$

Example 6.1.4
Find the scalar projection and vector projection of $\vec{u} =< 1, 1, 2 >$ onto $\vec{v} =< -2, 3, 1 >$.

Solution.
We have

$$\begin{aligned}\text{comp}_{\vec{v}}\vec{u} =& \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||} = \frac{1(-2) + (1)(3) + 2(1)}{\sqrt{(-2)^2 + 3^2 + 1^2}} = \frac{3}{\sqrt{14}}\\ \text{Proj}_{\vec{v}}\vec{u} =& \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}\frac{\vec{v}}{||\vec{v}||}\\ =& \frac{3}{14}\vec{v} = -\frac{2}{7}\vec{i} + \frac{9}{14}\vec{j} + \frac{3}{14}\vec{k} \blacksquare\end{aligned}$$

Applications
As pointed out earlier in the section, scalar products are used in Physics. For instance, in finding the work done by a force applied on an object.

Example 6.1.5
A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

Solution.
The work done is

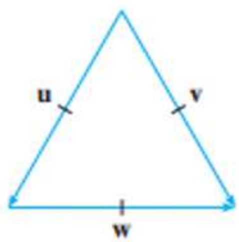
$$W = F \cdot d \cos 35^\circ = 70(100) \cos 35^\circ \approx 5734 \ J \blacksquare$$

Practice Problems

Problem 6.1.1
Find $\vec{a}\cdot\vec{b}$ where $\vec{a} =< 4, 1, \frac{1}{4} >$ and $\vec{b} =< 6, -3, -8 >$.

Problem 6.1.2
Find $\vec{a}\cdot\vec{b}$ where $||\vec{a}|| = 6,\ ||\vec{b}|| = 5$ and the angle between the two vectors is $120°$.

Problem 6.1.3
If $\vec{u}$ is a unit vector, find $\vec{u}\cdot\vec{v}$ and $\vec{u}\cdot\vec{w}$ using the figure below.



Problem 6.1.4
Find the angle between the vectors $\vec{a} =< 4, 3 >$ and $\vec{b} =< 2, -1 >$.

Problem 6.1.5
Find the angle between the vectors $\vec{a} =< 4, -3, 1 >$ and $\vec{b} =< 2, 0, -1 >$.

Problem 6.1.6
Determine whether the given vectors are orthogonal, parallel, or neither.
(a) $\vec{a} =< -5, 3, 7 >$ and $\vec{b} =< 6, -8, 2 >$.
(b) $\vec{a} =< 4, 6 >$ and $\vec{b} =< -3, 2 >$.
(c) $\vec{a} = -\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$.
(d) $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ and $\vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$.

Problem 6.1.7
Use vectors to decide whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.

Problem 6.1.8
Find a unit vector that is orthogonal to both $\vec{i} + \vec{j}$ and $\vec{i} + \vec{k}$.

Problem 6.1.9
Find the acute angle between the lines $2x - y = 3$ and $3x + y = 7$.

Problem 6.1.10
Find the scalar and vector projections of the vector $\vec{b} =< 1, 2, 3 >$ onto $\vec{a} =< 3, 6, -2 >$.

Problem 6.1.11
If $\vec{a} =< 3, 0, -1 >$, find a vector $\vec{b}$ such that $\text{comp}_{\vec{a}}\vec{b} = 2$.

Problem 6.1.12
Find the work done by a force $\vec{F} = 8\vec{i} - 6\vec{j} + 9\vec{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

Problem 6.1.13
A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of $40°$ above the horizontal moves the sled 80 ft. Find the work done by the force.

6.2 Directional Derivatives and the Gradient Vector

Given a function $z = f(x, y)$ and let (x_0, y_0) be in the domain of f. We wish to find the rate of change of f at (x_0, y_0) in the direction of a unit vector $\vec{u} =< a, b >$. To do this, we consider the vertical plane to the graph S of f that passes through the point $P(x_0, y_0, z_0)$ in the direction of $\vec{u}$. This plane inersects the graph S in a curve C. (See Figure 6.2.1.)

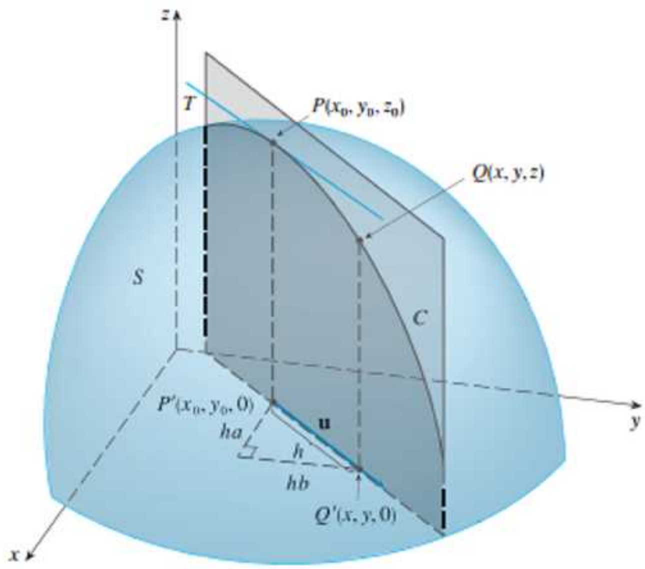


Figure 6.2.1

The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\vec{u}$. Let $Q(x, y, z)$ be an arbitrary point on C and let $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$ be the orthogonal projection of P and Q respectively onto the $xy-$plane. Then the vectors $\overrightarrow{P'Q'} =< x-x_0, y-y_0, 0 >$ is parallel to $\vec{u}$ so that $\overrightarrow{P'Q'} = h\vec{u}$ for some scalar h. Hence, $x = x_0+ha,\ y = y_0+hb$ and

$$\frac{f(x, y) - f(x_0, y_0)}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

If we take the limit of the above average rate as $h \to 0$, we obtain the rate of change of z(with respect to distance) in the direction of $\vec{u}$, which is called the **directional derivative** of f at (x_0, y_0) in the direction of $\vec{u}$. We write

$$f_{\vec{u}}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}.$$

Notice that if $\vec{u} = \vec{i}$ then $a = 1$ and $b = 0$ so that $f_{\vec{u}}(x_0, y_0) = f_x(x_0, y_0)$. That is, f_x is the rate of change of f in the $x-$ direction. Likewise, if $\vec{u} = \vec{j}$ then $a = 0$ and $b = 1$ so that $f_{\vec{u}}(x_0, y_0) = f_y(x_0, y_0)$.
The following theorem provides a formula for computing the directional derivative.

Theorem 6.2.1
If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} =< a, b >$ and

$$f_{\vec{u}}(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Proof.
Fix a point (x_0, y_0) in the domain of f and consider the single variable function $g(h) = f(x_0 + ha, y_0 + hb)$. Then

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x, y)}{h} = f_{\vec{u}}(x_0, y_0).$$

Let $x = x_0 + ah$ and $y = y_0 + bh$. Using the Chain Rule, we find

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$

Letting $h = 0$ in the above expression, we find

$$f_{\vec{u}}(x_0, y_0) = g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b \ \blacksquare \tag{6.2.1}$$

Example 6.2.1
Find $u_{\vec{v}}(4, 0)$ if $u(x, y) = x + y^2$ and $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

Solution.
We have

$$u_{\vec{v}}(4, 0) = u_x(4, 0)\left(\frac{1}{2}\right) + u_y(4, 0)\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2} \ \blacksquare$$

<u>**The Gradient Vector**</u>
The **gradient** is a generalization of the usual concept of derivative of a function of one variable to functions of several variables. For a function $u(x, y)$ or $u(x, y, z)$, the gradient are, respectively,

$$\nabla u(x,y) = u_x\vec{i} + u_y\vec{j} \text{ and } \nabla u(x,y,z) = u_x\vec{i} + u_y\vec{j} + u_z\vec{k}.$$

Example 6.2.2
Let $F(x,y,z) = u(x,y) - z$. Find $\nabla F(x,y,z)$.

Solution.
We have

$$\nabla F(x,y,z) = u_x\vec{i} + u_y\vec{j} - \vec{k} \blacksquare$$

Example 6.2.3
Find the gradient vector of $f(x,y,z) = (2x-3y+5z)^5$.

Solution.
We have

$$\begin{aligned} f_x(x,y,z) =& 10(2x-3y+5z)^4 \\ f_y(x,y,z) =& -15(2x-3y+5z)^4 \\ f_z(x,y,z) =& 25(2x-3y+5z)^4. \end{aligned}$$

Thus,

$$\nabla f(x,y,z) = 5(2x-3y+5z)^4[2\vec{i} - 3\vec{j} + 5\vec{k}] \blacksquare$$

With the notation for the gradient vector, we can rewrite the expression (6.2.1) for the directional derivative as

$$f_{\vec{u}}(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u}.$$

This expresses the directional derivative in the direction of $\vec{u}$ as the scalar projection of the gradient vector onto $\vec{u}$.

Maximizing the Directional Derivative

Suppose we have a function of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

Theorem 6.2.2
The maximum value of the directional derivative of a function $f(x,y)$ or $f(x,y,z)$ at a point (x,y) or (x,y,z) is $||\nabla f||$ and it occurs in the direction of the gradient of f at that point.

Proof.
We have

$$f_{\vec{u}}(x,y) = \nabla f \cdot \vec{u} = ||\nabla f||||\vec{u}|| \cos\theta = ||\nabla f|| \cos\theta,$$

where θ is the angle between ∇f and $\vec{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $f_{\vec{u}}$ is $||\nabla f||$ and it occurs when $\theta = 0$, that is, when $\vec{u}$ has the same direction as ∇f ■

Example 6.2.4
Find the maximum rate of change of the function $u(x,y) = 50 - x^2 - 2y^2$ at the point $(1,-1)$.

Solution.
The maximum rate of change occurs in the direction of the gradient vector:

$$\nabla u(1,-1) = u_x(1,-1)\vec{i} + u_y(1,-1)\vec{j} = -2\vec{i} + 4\vec{j}.$$

The maximum rate of change at $(1,-1)$ is

$$||\nabla u(1,-1)|| = \sqrt{(-2)^2 + 4^2} = 2\sqrt{5} \blacksquare$$

<u>**Significance of the Gradient Vector**</u>
Suppose that a curve in 3-D is defined parametrically by the equations $x = x(t),\ y = y(t),\ z = z(t)$, where t is a parameter. This curve can be described by the **vector function**

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

Its derivative is the tangent vector to the curve (See Figure 6.2.2) and is given by

$$\frac{d}{dt}(\vec{r}(t)) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}.$$

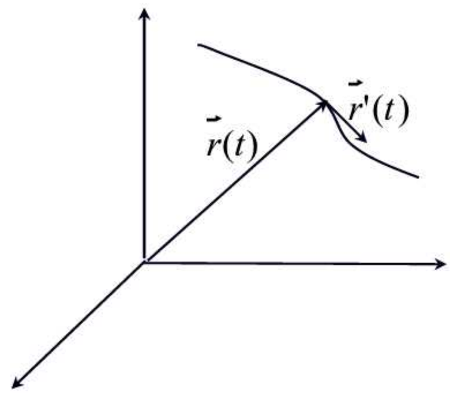


Figure 6.2.2

Now, for a function in two variables $u(x,y)$, the equation $u(x,y) = C$ is called a **level curve** of u(a **level surface** of $u(x,y,z)$). The level curves $u(x,y) = C$ are just the traces of the graph of $u(x,y)$ in the horizontal plane $z = C$ projected down to the $xy-$plane.
An important property of the gradient of u is that it is normal to a level surface of u at every point. To see this, let S be the level surface $f(x,y,z) = k$ and $P_0(x_0, y_0, z_0)$ be a point on S. Let C be any curve on S that passes through P_0. We can describe C in parametric form $x = x(t), y = y(t)$, and $z = z(t)$. Any point on C satisfies $f(x(t), y(t), z(t)) = k$. Differentiating both sides of this equation with respect to t we find by means of the Chain Rule

$$f_x(x,y,z)x'(t) + f_y(x,y,z)y'(t) + f_z(x,y,z)z'(t) = 0$$

which can be written as $\nabla f \cdot \vec{r'(t)} = 0$. This means that the gradient is normal to a level surface (respectively a level curve). See Figure 6.2.3.

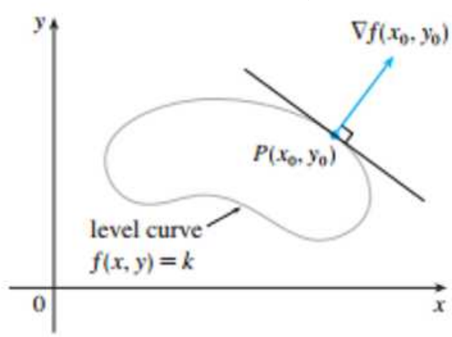


Figure 6.2.3

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 6.2.4 by making it perpendicular to all of the contour lines.

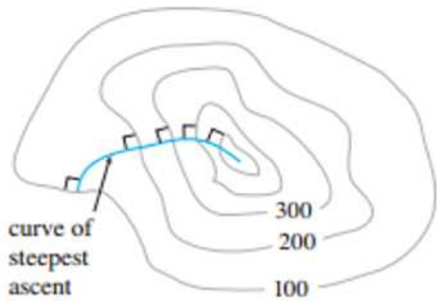


Figure 6.2.4

Vector Fields and Integral Curves

In vector calculus, a **vector field** is a function $\vec{F}(x, y)$ (or $\vec{F}(x, y, z)$ in 3-D space) that assigns a vector to each point of its domain as shown in Figure 6.2.5.

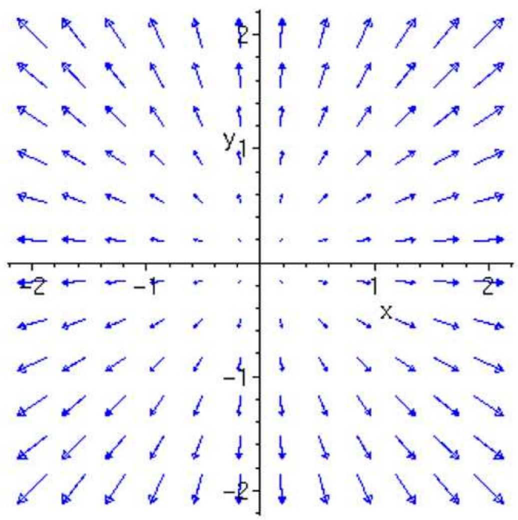


Figure 6.2.5

Creating vector fields manually is very tedious. Thus, vector fields are generally generated using computer softwares such as Mathematica, Maple, or Mathlab.

Example 6.2.5
The gradient vector of a function is an example of a vector field called the **gradient vector field**. Sketch the gradient vector field of the function

$$u(x,y) = x^2 + y^2.$$

Describe the level curves of $u(x,y)$.

Solution.
The gradient vector field of the given function is

$$\nabla u(x,y) = 2x\vec{i} + 2y\vec{j}.$$

A level curve is defined by the equation

$$x^2 + y^2 = C, \ C \geq 0.$$

Thus, level curves are circles centered at the origin. Figure 6.2.6 shows the gradient vector field as well as some of the level curves.

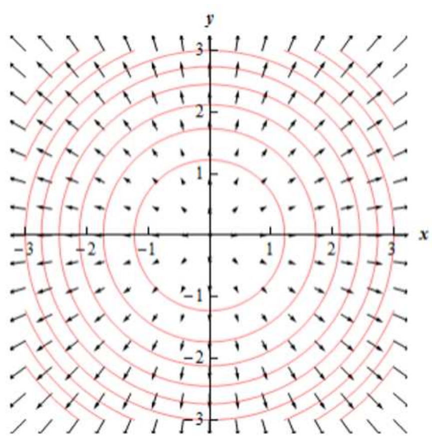


Figure 6.2.6

For example, at the point $(1, 2)$, the corresponding vector in the vector field is the vector with tail $(1, 2)$ and tip $(2, 4)$ ■

An **integral curve** of a vector field is a smooth curve[2] Γ such that $\vec{F}(x, y)$ assigns a tangent vector at each point of Γ. For example, the integral curves of the vector field $\vec{F}(x, y) = y\vec{i} - x\vec{j}$ are circles centered at the origin. See Figure 6.2.7.

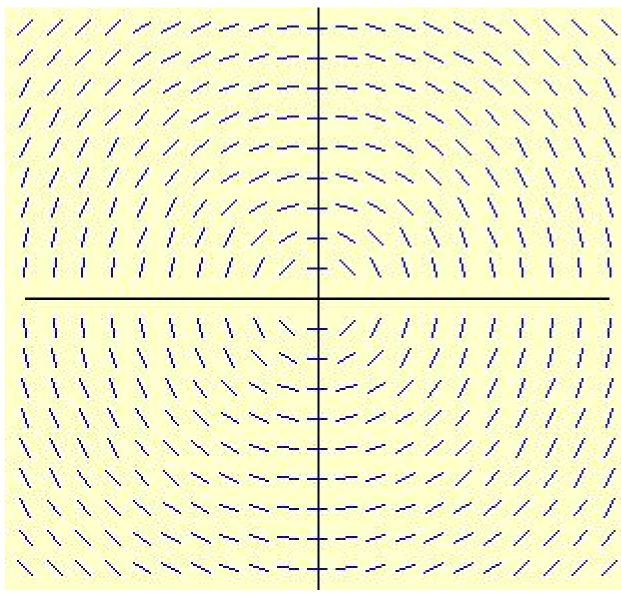


Figure 6.2.7

[2]If $\vec{r}(t)$ is a parametrization of Γ then $\vec{r'}(t)$ is continuous and $\vec{r'}(t) \neq \vec{0}$.

Practice Problems

Problem 6.2.1
Find the gradient of the function

$$F(x,y,z) = e^{xyz} + \sin(xy).$$

Problem 6.2.2
Find the gradient of the function

$$F(x,y,z) = x\cos\left(\frac{y}{z}\right).$$

Problem 6.2.3
Describe the level surfaces of the function $f(x,y,z) = (x-2)^2 + (y-3)^2 + (z+5)^2$.

Problem 6.2.4
Find the directional derivative of $u(x,y) = 4x^2 + y^2$ in the direction of $\vec{a} = \vec{i} + 2\vec{j}$ at the point $(1,1)$.

Problem 6.2.5
Find the directional derivative of $u(x,y,z) = x^2z + y^3z^2 - xyz$ in the direction of $\vec{a} = -\vec{i} + 3\vec{k}$ at the point (x,y,z).

Problem 6.2.6
Find the maximum rate of change of the function $u(x,y) = ye^{xy}$ at the point $(0,2)$ and the direction in which this maximum occurs.

Problem 6.2.7
Find the gradient vector field for the function $u(x,y,z) = e^z - \ln(x^2 + y^2)$.