First Order Partial Differential Equations

Many problems in the mathematical, physical, and engineering sciences deal with the formulation and the solution of first order partial differential equations. Our first task is to understand simple first order equations. In applications, first order partial differential equations are most commonly used to describe dynamical processes, and so time, t, is one of the independent variables. Most of our discussion will focus on dynamical models in a single space dimension, bearing in mind that most of the methods can be readily extended to higher dimensional situations. First order partial differential equations and systems model a wide variety of wave phenomena, including transport of solvents in fluids, flood waves, acoustics, gas dynamics, glacier motion, traffic flow, and also a variety of biological and ecological systems. From a mathematical point of view, first order partial differential equations have the advantage of providing conceptual basis that can be utilized in the study of higher order partial differential equations.
In this chapter we introduce the basic definitions of first order partial differential equations. We then derive the one dimensional spatial transport eqution and discuss some methods of solutions. One general method of solvability for quasilinear first order partial differential equation, known as the method of characteristics, is analyzed.

5 Classification of First Order PDEs

In this section, we present the basic definitions pertained to first order PDE. By a **first order partial differential equation** in two variables x and y we mean any equation of the form

$$F(x, y, u, u_x, u_y) = 0. \tag{5.1}$$

In what follows the functions $a, b,$ and c are assumed to be continuously differentiable functions. If Equation (5.1) can be written in the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{5.2}$$

then we say that the equation is **quasi-linear**. The following are examples of quasi-linear equations:

$$uu_x + u_y + cu^2 = 0$$

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u.$$

If Equation (5.1) can be written in the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \tag{5.3}$$

then we say that the equation is **semi-linear**. The following are examples of semi-linear equations:

$$xu_x + yu_y = u^2 + x^2$$

$$(x + 1)^2 u_x + (y - 1)^2 u_y = (x + y)u^2.$$

If Equation (5.1) can be written in the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y) \tag{5.4}$$

then we say that the equation is **linear**. Examples of linear equations are:

$$xu_x + yu_y = cu$$

$$(y - z)u_x + (z - x)u_y + (x - y)u_z = 0.$$

A first order pde that is not linear is said to be **non-linear.** Examples of non-linear equations are:

$$u_x + cu_y^2 = xy$$

$$u_x^2 + u_y^2 = c.$$

First order partial differential equations are classified as either linear or nonlinear. Clearly, linear equations are a special kind of quasi-linear equation (5.2) if a and b are functions of x and y only and c is a linear function of u. Likewise, semi-linear equations are quasilinear equations if a and b are functions of x and y only. Also, semi-linear equations (5.3) reduces to a linear equation if c is linear in u.

A linear first order partial differential equation is called **homogeneous** if $d(x,y) \equiv 0$ and **non-homogeneous** if $d(x,y) \neq 0$. Examples of linear homogeneous equations are:

$$xu_x + yu_y = cu$$

$$(y-z)u_x + (z-x)u_y + (x-y)u_z = 0.$$

Examples of non-homogeneous equations are:

$$u_x + (x+y)u_y - u = e^x$$

$$yu_x + xu_y = xy.$$

Recall that for an ordinary linear differential equation, the general solution depends mainly on arbitrary constants. Unlike ODEs, in linear partial differential equations, the general solution depends on arbitrary functions.

Example 5.1

Solve the equation $u_t(x,t) = 0$.

Solution.

The general solution is given by $u(x,t) = f(x)$ where f is an arbitrary differentiable function of x ■

Example 5.2

Consider the transport equation

$$au_t(x,t) + bu_x(x,t) = 0$$

where a and b are constants. Show that $u(x,t) = f(bt - ax)$ is a solution to the given equation, where f is an arbitrary differentiable function in one variable.

Solution.

Let $v(x,t) = bt - ax$. Using the chain rule we see that $u_t(x,t) = bf_v(v)$ and $u_x(x,t) = -af_v(v)$. Hence, $au_t(x,t) + bu_x(x,t) = abf_v(v) - abf_v(v) = 0$ ■

Practice Problems

Problem 5.1
Classify each of the following PDE as linear, quasi-linear, semi-linear, or non-linear.
(a) $xu_x + yu_y = \sin(xy)$.
(b) $u_t + uu_x = 0$
(c) $u_x^2 + u^3u_y^4 = 0$.
(d) $(x+3)u_x + xy^2u_y = u^3$.

Problem 5.2
Show that $u(x,y) = e^x f(2x-y)$, where f is a differentiable function of one variable, is a solution to the equation

$$u_x + 2u_y - u = 0.$$

Problem 5.3
Show that $u(x,y) = x\sqrt{xy}$ satisfies the equation

$$xu_x - yu_y = u$$

subject to

$$u(y,y) = y^2, \ y \geq 0.$$

Problem 5.4
Show that $u(x,y) = \cos(x^2+y^2)$ satisfies the equation

$$-yu_x + xu_y = 0$$

subject to

$$u(0,y) = \cos y^2.$$

Problem 5.5
Show that $u(x,y) = y - \frac{1}{2}(x^2-y^2)$ satisfies the equation

$$\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{y}$$

subject to $u(x,1) = \frac{1}{2}(3-x^2)$.

Problem 5.6
Find a relationship between a and b if $u(x,y) = f(ax+by)$ is a solution to the equation $3u_x - 7u_y = 0$ for any differentiable function f such that $f'(x) \neq 0$ for all x.

Problem 5.7
Reduce the partial differential equation

$$au_x + bu_y + cu = 0$$

to a first order ODE by introducing the change of variables $s = bx - ay$ and $t = x$.

Problem 5.8
Solve the partial differential equation

$$u_x + u_y = 1$$

by introducing the change of variables $s = x - y$ and $t = x$.

Problem 5.9
Show that $u(x,y) = e^{-4x}f(2x - 3y)$ is a solution to the first-order PDE

$$3u_x + 2u_y + 12u = 0.$$

Problem 5.10
Derive the general solution of the PDE

$$au_t + bu_x = u, \quad b \neq 0$$

by using the change of variables $v = ax - bt$ and $w = x$.

Problem 5.11
Derive the general solution of the PDE

$$au_x + bu_y = 0, \quad a \neq 0$$

by using the change of variables $s = bx - ay$ and $t = x$.

Problem 5.12
Write the equation

$$u_t + cu_x + \lambda u = f(x,t),\ c \neq 0$$

in the coordinates $v = x - ct,\ w = x$.

Problem 5.13
Suppose that $u(x,t) = w(x - ct)$ is a solution to the PDE

$$xu_x + tu_t = Au$$

where A and c are constants. Let $v = x - ct$. Write the differential equation with unknown function $w(v)$.