4 The Method of Separation of Variables for ODEs

The method of separation of variables that you have seen in the theory of ordinary differential equations has an analogue in the theory of partial differential equations (Section 17). In this section, we review the method for ordinary differentiable equations.
A first order differential equation is **separable** if it can be written with one variable only on the left and the other variable only on the right:

$$f(y)y' = g(t).$$

To solve this equation, we proceed as follows. Let $F(t)$ be an antiderivative of $f(t)$ and $G(t)$ be an antiderivative of $g(t)$. Then by the Chain Rule

$$\frac{d}{dt}F(y) = \frac{dF}{dy}\frac{dy}{dt} = f(y)y'.$$

Thus,

$$0 = f(y)y' - g(t) = \frac{d}{dt}F(y) - \frac{d}{dt}G(t) = \frac{d}{dt}[F(y) - G(t)].$$

It follows that

$$F(y) - G(t) = C$$

which is equivalent to

$$\int f(y)y'dt = \int g(t)dt + C.$$

As you can see, the result is generally an implicit equation involving a function of y and a function of t. It may or may not be possible to solve this to get y explicitly as a function of t. For an initial value problem, substitute the values of t and y by t_0 and y_0 to get the value of C.

Remark 4.1
If F is a differentiable function of y and y is a differentiable function of t and both F and y are given then the chain rule allows us to find $\frac{dF}{dt}$ given by

$$\frac{dF}{dt} = \frac{dF}{dy} \cdot \frac{dy}{dt}.$$

For separable equations, we are given $f(y)y' = \frac{dF}{dt}$ and we are asked to find $F(y)$. This process is referred to as "reversing the chain rule."

Example 4.1
Solve the initial value problem $y' = 6ty^2, \quad y(1) = \frac{1}{25}$.

Solution.
Separating the variables and integrating both sides we obtain

$$\int \frac{y'}{y^2} dt = \int 6t dt$$

or

$$-\int \frac{d}{dt}\left(\frac{1}{y}\right) dt = \int 6t dt.$$

Thus,

$$-\frac{1}{y(t)} = 3t^2 + C.$$

Since $y(1) = \frac{1}{25}$, we find $C = -28$. The unique solution to the IVP is then given explicitly by

$$y(t) = \frac{1}{28 - 3t^2} \blacksquare$$

Example 4.2
Solve the IVP $yy' = 4\sin(2t), \quad y(0) = 1$.

Solution.
This is a separable differential equation. Integrating both sides we find

$$\int \frac{d}{dt}\left(\frac{y^2}{2}\right) dt = 4\int \sin(2t) dt.$$

Thus,

$$y^2 = -4\cos(2t) + C.$$

Since $y(0) = 1$, we find $C = 5$. Now, solving explicitly for $y(t)$ we find

$$y(t) = \pm\sqrt{-4\cos t + 5}.$$

Since $y(0) = 1$, we have $y(t) = \sqrt{-4\cos t + 5}$. The interval of existence of the solution is the interval $-\infty < t < \infty$ $\blacksquare$

Practice Problems

Problem 4.1
Solve the (separable) differential equation

$$y' = te^{t^2 - \ln y^2}.$$

Problem 4.2
Solve the (separable) differential equation

$$y' = \frac{t^2y - 4y}{t + 2}.$$

Problem 4.3
Solve the (separable) differential equation

$$ty' = 2(y - 4).$$

Problem 4.4
Solve the (separable) differential equation

$$y' = 2y(2 - y).$$

Problem 4.5
Solve the IVP

$$y' = \frac{4\sin(2t)}{y}, \quad y(0) = 1.$$

Problem 4.6
Solve the IVP:

$$yy' = \sin t, \quad y(\frac{\pi}{2}) = -2.$$

Problem 4.7
Solve the IVP:

$$y' + y + 1 = 0, \quad y(1) = 0.$$

Problem 4.8
Solve the IVP:

$$y' - ty^3 = 0, \quad y(0) = 2.$$

Problem 4.9
Solve the IVP:
$$y' = 1 + y^2, \quad y(\frac{\pi}{4}) = -1.$$

Problem 4.10
Solve the IVP:
$$y' = t - ty^2, \quad y(0) = \frac{1}{2}.$$

Problem 4.11
Solve the equation $3u_y + u_{xy} = 0$ by using the substitution $v = u_y$.

Problem 4.12
Solve the IVP
$$(2y - \sin y)y' = \sin t - t, \quad y(0) = 0.$$

Problem 4.13
State an initial value problem, with initial condition imposed at $t_0 = 2$, having implicit solution $y^3 + t^2 + \sin y = 4$.

Problem 4.14
Can the differential equation
$$\frac{dy}{dx} = x^2 - xy$$
be solved by the method of separation of variables? Explain.