Review of Some ODE Results

Later on in this book, we will encounter problems where a given partial differential equation is reduced to an ordinary differential equation by means of a given change of variables. Then techniques from the theory of ODE are required in solving the transformed ODE. In this chapter, we include some of the results from ODE theory that will be needed in our future discussions.

3 The Method of Integrating Factor

In this section, we discuss a technique for solving the first order linear non-homogeneous equation

$$y' + p(t)y = g(t) \tag{3.1}$$

where $p(t)$ and $g(t)$ are continuous on the open interval $a < t < b$.
Since $p(t)$ is continuous, it has an antiderivative namely $\int p(t)dt$. Let $\mu(t) = e^{\int p(t)dt}$. Multiply Equation (3.1) by $\mu(t)$ and notice that the left hand side of the resulting equation is the derivative of a product. Indeed,

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t).$$

Integrate both sides of the last equation with respect to t to obtain

$$\mu(t)y = \int \mu(t)g(t)dt + C.$$

Hence,

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)dt + \frac{C}{\mu(t)}$$

or

$$y(t) = e^{-\int p(t)dt}\int e^{\int p(t)dt}g(t)dt + Ce^{-\int p(t)dt}.$$

Notice that the second term of the previous expression is just the general solution for the homogeneous equation

$$y' + p(t)y = 0$$

whereas the first term is a solution to the non-homogeneous equation. That is, the general solution to Equation (3.1) is the sum of a particular solution of the non-homogeneous equation and the general solution of the homogeneous equation.

Example 3.1
Solve the initial value problem

$$y' - \frac{y}{t} = 4t, \quad y(1) = 5.$$

Solution.
We have $p(t) = -\frac{1}{t}$ so that $\mu(t) = \frac{1}{t}$. Multiplying the given equation by the integrating factor and using the product rule we notice that

$$\left(\frac{1}{t}y\right)' = 4.$$

Integrating with respect to t and then solving for y we find that the general solution is given by

$$y(t) = t\int 4dt + Ct = 4t^2 + Ct.$$

Since $y(1) = 5$, we find $C = 1$ and hence the unique solution to the IVP is $y(t) = 4t^2 + t,\ 0 < t < \infty$ ■

Example 3.2
Find the general solution to the equation

$$y' + \frac{2}{t}y = \ln t,\ t > 0.$$

Solution.
The integrating factor is $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$. Multiplying the given equation by t^2 to obtain

$$(t^2y)' = t^2 \ln t.$$

Integrating with respect to t we find

$$t^2y = \int t^2 \ln t dt + C.$$

The integral on the right-hand side is evaluated using integration by parts with $u = \ln t, dv = t^2dt, du = \frac{dt}{t}, v = \frac{t^3}{3}$ obtaining

$$t^2y = \frac{t^3}{3}\ln t - \frac{t^3}{9} + C$$

Thus,

$$y = \frac{t}{3}\ln t - \frac{t}{9} + \frac{C}{t^2}$$ ■

Example 3.3
Solve

$$au_x + bu_y + cu = 0$$

by using the change of variables $s = ax + by$ and $t = bx - ay$.

Solution.
By the Chain rule for functions of two variables, we have

$$\begin{aligned} u_x =& u_s s_x + u_t t_x = au_s + bu_t \\ u_y =& u_s s_y + u_t t_y = bu_s - au_t. \end{aligned}$$

Substituting into the given equation, we find

$$u_s + \frac{c}{a^2 + b^2} u = 0.$$

Solving this equation using the integrating factor method we find

$$u(s, t) = f(t)e^{-\frac{cs}{a^2+b^2}}$$

where f is an arbitrary differentiable function of f. Switching back to x and y we obtain

$$u(x, y) = f(bx - ay)e^{-\frac{c}{a^2+b^2}(ax+by)} \blacksquare$$

Practice Problems

Problem 3.1
Solve the IVP: $y' + 2ty = t, \quad y(0) = 0.$

Problem 3.2
Find the general solution: $y' + 3y = t + e^{-2t}$.

Problem 3.3
Find the general solution: $y' + \frac{1}{t}y = 3\cos t, \ t > 0.$

Problem 3.4
Find the general solution: $y' + 2y = \cos(3t)$.

Problem 3.5
Find the general solution: $y' + (\cos t)y = -3\cos t$.

Problem 3.6
Given that the solution to the IVP $ty' + 4y = \alpha t^2, \ y(1) = -\frac{1}{3}$ exists on the interval $-\infty < t < \infty$. What is the value of the constant α?

Problem 3.7
Suppose that $y(t) = Ce^{-2t} + t + 1$ is the general solution to the equation $y' + p(t)y = g(t)$. Determine the functions $p(t)$ and $g(t)$.

Problem 3.8
Suppose that $y(t) = -2e^{-t} + e^t + \sin t$ is the unique solution to the IVP $y' + y = g(t), \ y(0) = y_0$. Determine the constant y_0 and the function $g(t)$.

Problem 3.9
Find the value (if any) of the unique solution to the IVP $y' + (1 + \cos t)y = 1 + \cos t, \ y(0) = 3$ in the long run?

Problem 3.10
Solve the initial value problem $ty' = y + t, \quad y(1) = 7.$

Problem 3.11
Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$. Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

Problem 3.12
Solve the initial-value problem $y' + y = e^t y^2, y(0) = 1$ using the substitution $u(t) = \frac{1}{y(t)}$

Problem 3.13
Solve the initial-value problem $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$

Problem 3.14
Solve $y' - \frac{1}{t}y = \sin t, \quad y(1) = 3$. Express your answer in terms of the **sine integral**, $Si(t) = \int_0^t \frac{\sin s}{s} ds$.