25 Applications of Fourier Transforms to PDEs

Fourier transform is a useful tool for solving differential equations. In this section, we apply Fourier transforms in solving various PDE problems. Contrary to Laplace transform, which usually uses the time variable, the Fourier transform is applied to the spatial variable on the whole real line.
The Fourier transform will be applied to the spatial variable x while the variable t remains fixed. The PDE in the two variables x and t passes under the Fourier transform to an ODE in the $t-$variable. We solve this ODE to obtain the transformed solution $\hat{u}$ which can be converted to the original solution u by means of the inverse Fourier transform. We illustrate these ideas in the examples below.

First Order Transport Equation
Consider the initial value problem

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x).$$

Let $\hat{u}(\xi, t)$ be the Fourier transform of u in x. Performing the Fourier transform on both the PDE and the initial condition, we reduce the PDE into an ODE in t

$$\frac{\partial \hat{u}}{\partial t} + i\xi c\hat{u} = 0$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

Solution of the ODE gives

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-i\xi ct}.$$

Thus,

$$u(x, t) = \mathcal{F}^{-1}[\hat{u}(\xi, t)] = f(x - ct)$$

which is exactly the same as we obtained by using the method of characteristics.(Section 8)

Second Order Wave Equation
Consider the two dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \; x \in \mathbb{R}, \; t > 0$$

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x).$$

Again, by performing the Fourier transform of u in x, we reduce the PDE problem into an ODE problem in the variable t:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2\xi^2\hat{u}$$

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$
$$\hat{u}_t(\xi,0) = \hat{g}(\xi).$$

General solution to the ODE is

$$\hat{u}(\xi,t) = \Phi(\xi)e^{-i\xi ct} + \Psi(\xi)e^{i\xi ct}$$

where Φ and Ψ are two arbitrary functions of ξ. Performing the inverse transformation and making use of the translation theorem, we get the general solution

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

where $\mathcal{F}(\phi) = \Phi$ and $\mathcal{F}(\psi) = \Psi$. But

$$\Phi(\xi) = \frac{1}{2}\left[\hat{f}(\xi) - \frac{1}{i\xi c}\hat{g}(\xi)\right]$$

$$\Psi(\xi) = \frac{1}{2}\left[\hat{f}(\xi) + \frac{1}{i\xi c}\hat{g}(\xi)\right].$$

By using the integration property, we find the inverse transforms of Φ and Ψ

$$\phi(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_0^x g(s)ds\right]$$

$$\psi(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_0^x g(s)ds\right].$$

Application of the translation property then yields directly the D'Alambert solution

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] - \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Second Order Heat Equation
Next, we consider the heat equation

$$u_t = ku_{xx}, \quad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = f(x).$$

Performing Fourier Transform in x for the PDE and the initial condition, we obtain

$$\frac{\partial \hat{u}}{\partial t} = -k\xi^2 \hat{u}$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

Treating ξ as a parameter, we obtain the solution to the above ODE problem

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-k\xi^2 t}.$$

Application of the convolution theorem yields

$$\begin{aligned} u(x,t) =& f(x) * \mathcal{F}^{-1}[e^{-k\xi^2 t}] \\ =& f(x) * \left[\frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}\right] \\ =& \frac{1}{\sqrt{4\pi kt}} \int_0^x f(s) e^{-\frac{(x-s)^2}{4kt}} ds. \end{aligned}$$

Laplace's Equation in 2D
Consider the problem

$$\Delta u = u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ 0 < y < L$$

$$u(x, 0) = 0$$

$$u(x, L) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise.} \end{cases}$$

Performing Fourier Transform in x for the PDE we obtain the second order ODE in y

$$\hat{u}_{yy} = \xi^2 \hat{u}.$$

The general solution is given by

$$\hat{u}(\xi, y) = A(\xi)\sinh(\xi y) + B(\xi)\cosh(\xi y).$$

Using the boundary condition $\hat{u}(\xi,0)=0$ we find $B(\xi)=0$. Using the second boundary condition we find

$$\begin{aligned}\hat{u}(\xi,L)=&\int_{-\infty}^{\infty}u(x,L)e^{-i\xi x}dx\\=&\int_{-a}^{a}e^{-i\xi x}dx=\int_{-a}^{a}\cos(\xi x)dx\\=&\frac{2\sin(\xi a)}{\xi}.\end{aligned}$$

Hence,

$$A(\xi)\sinh(\xi L)=\frac{2\sin(\xi a)}{\xi}$$

and this implies

$$A(\xi)=\frac{2\sin(\xi a)}{\xi\sinh(\xi L)}.$$

Thus,

$$\hat{u}(\xi,y)=\frac{2\sin(\xi a)}{\xi\sinh(\xi L)}\sinh(\xi y).$$

Taking inverse Fourier transform we find

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin(\xi a)}{\xi\sinh(\xi L)}\sinh(\xi y)e^{i\xi x}d\xi.$$

Using Euler's formula , and the fact that

$$\frac{2\sin(\xi a)}{\xi\sinh(\xi L)}\sinh(\xi y)\sin(\xi x)$$

is odd in ξ, we arrive at

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin(\xi a)}{\xi\sinh(\xi L)}\sinh(\xi y)\cos(\xi x)d\xi\ \blacksquare$$

Practice Problems

Problem 25.1
Solve, by using Fourier transform

$$u_t + cu_x = 0$$
$$u(x,0) = e^{-\frac{x^2}{4}}.$$

Problem 25.2
Solve, by using Fourier transform

$$u_t = ku_{xx} - \alpha u, \quad x \in \mathbb{R}$$
$$u(x,0) = e^{-\frac{x^2}{\gamma}}.$$

Problem 25.3
Solve the heat equation

$$u_t = ku_{xx}$$

subject to the initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 25.4
Use Fourier transform to solve the heat equation

$$u_t = u_{xx} + u, \quad -\infty < x < \infty < \quad t > 0$$
$$u(x,0) = f(x).$$

Problem 25.5
Prove that

$$\int_{-\infty}^{\infty} e^{-|\xi|y} e^{i\xi x} d\xi = \frac{2y}{x^2 + y^2}.$$

Problem 25.6
Solve the Laplace's equation in the half plane

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \ 0 < y < \infty$$

subject to the boundary condition

$$u(x,0) = f(x), \ |u(x,y)| < \infty.$$

Problem 25.7
Use Fourier transform to find the transformed equation of

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

where $\alpha, \beta > 0$.

Problem 25.8
Solve the initial value problem

$$u_t + 3u_x = 0$$
$$u(x, 0) = e^{-x}$$

using the Fourier transform.

Problem 25.9
Solve the initial value problem

$$u_t = ku_{xx}$$
$$u(x, 0) = e^{-x}$$

using the Fourier transform.

Problem 25.10
Solve the initial value problem

$$u_t = ku_{xx}$$
$$u(x, 0) = e^{-x^2}$$

using the Fourier transform.

Problem 25.11
Solve the initial value problem

$$u_t + cu_x = 0$$
$$u(x, 0) = x^2$$

using the Fourier transform.

Problem 25.12
Solve, by using Fourier transform

$$\Delta u = 0$$
$$u_y(x, 0) = f(x)$$
$$\lim_{x^2+y^2 \to \infty} u(x, y) = 0.$$