24 An introduction to Fourier Transforms

One of the problems with the theory of Fourier series discussed so far is that it applies only to periodic functions. There are many times when one would like to divide a function which is not periodic into a superposition of sines and cosines. The Fourier transform is the tool often used for this purpose. Like the Laplace transform, the Fourier transform is often an effective tool in finding explicit solutions to partial differential equations.

We will introduce the Fourier transform of $f(x)$ as a limiting case of a Fourier series. This requires a tedious discussion which we omit and rather explain the underlying ideas. More specifically, the approach we introduce is to construct Fourier series of $f(x)$ on progressively longer and longer intervals, and then take the limit as their lengths go to infinity. This limiting process converts the Fourier sums into integrals, and the resulting representation of a function is renamed the Fourier transform.

To start with, let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function with the properties $\lim_{x\to\pm\infty} f(x) = 0$ and $\int_0^\infty |f(x)|dx < \infty$. Define the function f_L which is equal to f in an interval of the form $[-\pi L, \pi L]$ and vanishes outside this interval. Note that $f(x) = \lim_{L\to\infty} f_L(x)$. This function can be extended to a periodic function, denoted by f_e, of period $2\pi T$ with $T > L$ and where $f_e(x) = f(x)$ for $|x| \le \pi L$ and 0 for $-\pi T \le x \le -\pi L$ and $\pi L \le x \le \pi T$. Note that $f(x) = \lim_{L\to\infty} f_L(x) = \lim_{L\to\infty} f_e(x)$. From the previous section we can find the complex Fourier series of f_e to be

$$f_e(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{inx}{T}} \tag{24.1}$$

where

$$c_n = \frac{1}{2\pi T}\int_{-\pi T}^{\pi T} f_e(x) e^{-\frac{inx}{T}} dx.$$

Let $\xi \in \mathbb{R}$. Multiply both sides of (24.1) by $e^{-i\xi x}$ and then integrate both sides from $-\pi T$ to πT. Assuming integration and summation can be interchanged we find

$$\int_{-\pi T}^{\pi T} f_e(x) e^{-i\xi x} dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi T}^{\pi T} e^{-i\xi x} e^{\frac{inx}{T}} dx.$$

It can be shown that the RHS converges, say to $\hat{f}(\xi)$, as $L \to \infty$ (and $T \to \infty$) Hence, we find

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx. \tag{24.2}$$

The function $\hat{f}$ is called the **Fourier transform** of f. We will use the notation $\mathcal{F}[f(x)] = \hat{f}(\xi)$.
Next, it can be shown that

$$\hat{f}\left(\frac{n}{T}\right) = 2\pi T c_n$$

so that

$$f_e(x) = \frac{1}{2\pi T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{T}\right) e^{\frac{inx}{T}}.$$

It can be shown that as $L \to \infty$, we have

$$\lim_{T\to\infty} \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{T}\right) e^{\frac{inx}{T}} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi$$

so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi \tag{24.3}$$

Equation (24.3) is called the **Fourier inversion formula** and we use the notation $\mathcal{F}^{-1}[\hat{f}(\xi)]$. Now, if we make use of Euler's formula, we can write the Fourier inversion formula in terms of sines and cosines,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \cos \xi x d\xi + \frac{i}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \sin \xi x d\xi$$

a superposition of sines and cosines of various frequencies.

Example 24.1
Find the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} e^{-ax} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

for some $a > 0$.

Solution.
We have

$$\begin{aligned}\hat{f}(\xi) =& \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \int_0^{\infty} e^{-ax}e^{-i\xi x}dx\\ =& \int_0^{\infty} e^{-ax-i\xi x}dx = \left.\frac{e^{-x(a+i\xi)}}{-(a+i\xi)}\right|_0^{\infty}\\ =& \frac{1}{a+i\xi} \blacksquare\end{aligned}$$

The following theorem lists the basic properties of the Fourier transform

Theorem 24.1
Let f, g, be piecewise continuous functions. Then we have the following properties:
(1) **Linearity:** $\mathcal{F}[\alpha f(x) + \beta g(x)] = \alpha\mathcal{F}[f(x)] + \beta\mathcal{F}[g(x)]$, where α and β are arbitrary numbers.
(2) **Shifting**: $\mathcal{F}[f(x-\alpha)] = e^{-i\alpha\xi}\mathcal{F}[f(x)]$.
(3) **Scaling:** $\mathcal{F}[f\left(\frac{x}{\alpha}\right)] = \alpha\mathcal{F}[f(\alpha x)]$.
(4) **Continuity:** If $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ then $\hat{f}$ is continuous in ξ.
(5) **Differentiation:** $\mathcal{F}[f^{(n)}(x)] = (i\xi)^n\mathcal{F}[f(x)]$.
(6) **Integration:** $\mathcal{F}\left[\int_0^x f(s)ds\right] = -\frac{1}{i\xi}\mathcal{F}[f(x)]$.
(7) **Parseval's Relation:** $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$.
(8) **Duality:** $\mathcal{F}[\mathcal{F}[f(x)]] = 2\pi f(-x)$.
(9) **Multiplication by** x^n : $\mathcal{F}[x^n f(x)] = i^n \hat{f}^{(n)}(\xi)$.
(10) **Gaussians:** $\mathcal{F}[e^{-\alpha x^2}] = \sqrt{\frac{\pi}{\alpha}}e^{-\frac{\xi^2}{4\alpha}}$.
(11) **Product:** $\mathcal{F}[(f(x)g(x)] = \frac{1}{2\pi}\mathcal{F}[f(x)] * \mathcal{F}[g(x)]$.
(12) **Convolution:** $\mathcal{F}[(f*g)(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)]$.

Example 24.2
Determine the Fourier transform of the Gaussian $u(x) = e^{-\alpha x^2},\ \alpha > 0$.

Solution.
We have

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-\alpha x^2}e^{-i\xi x}dx.$$

If we differentiate this relation with respect to the variable ξ and then integrate by parts we obtain

$$\begin{aligned}\hat{u}'(\xi) =& -i\int_{-\infty}^{\infty} xe^{-\alpha x^2}e^{-i\xi x}dx\\ =&\frac{i}{2\alpha}\int_{-\infty}^{\infty}\frac{d}{dx}(e^{-\alpha x^2})e^{-i\xi x}dx\\ =&\frac{i}{2\alpha}\left[e^{-\alpha x^2-i\xi x}\Big|_{-\infty}^{\infty}+i\xi\int_{-\infty}^{\infty}(e^{-\alpha x^2})e^{-i\xi x}dx\right]\\ =&\frac{i^2\xi}{2\alpha}\int_{-\infty}^{\infty}(e^{-\alpha x^2})e^{-i\xi x}dx=-\frac{\xi}{2\alpha}\hat{u}(\xi)\end{aligned}$$

Thus, we have arrived at the ODE $\hat{u}'(\xi) = -\frac{\xi}{2\alpha}\hat{u}(\xi)$ whose general solution has the form

$$\hat{u}(\xi) = Ce^{-\frac{\xi^2}{4\alpha}}.$$

Using a result from real analysis which states that

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi},$$

we can write

$$\hat{u}(0) = \int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}} = C,$$

and therefore

$$\hat{u}(\xi) = \sqrt{\frac{\pi}{\alpha}}e^{-\frac{\xi^2}{4\alpha}} \blacksquare$$

Example 24.3
Prove

$$\mathcal{F}[f(-x)] = \hat{f}(-\xi).$$

Solution.
Using a change of variables we find

$$\mathcal{F}[f(-x)] = \int_{-\infty}^{\infty} f(-x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = \hat{f}(-\xi) \blacksquare$$

Example 24.4
Prove

$$\mathcal{F}[\mathcal{F}[f(x)]] = 2\pi f(-x).$$

Solution.
We have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}d\xi$$

Thus,

$$2\pi f(-x) = \int_{-\infty}^{\infty}\hat{f}(\xi)e^{-i\xi x}d\xi = \mathcal{F}[\hat{f}(\xi)] = \mathcal{F}[\mathcal{F}[f(x)]] \blacksquare$$

The following theorem lists the properties of inverse Fourier transform

Theorem 24.2
Let f and g be piecewise continuous functions.
(1′) **Linearity:** $\mathcal{F}^{-1}[\alpha\hat{f}(\xi) + \beta\hat{g}(\xi)] = \alpha\mathcal{F}^{-1}[\hat{f}(\xi)] + \beta\mathcal{F}^{-1}[\hat{g}(\xi)]$.
(2′) **Derivatives:** $\mathcal{F}^{-1}[\hat{f}^{(n)}(\xi)] = (-ix)^n f(x)$.
(3′) **Multiplication by** ξ^n : $\mathcal{F}^{-1}[\xi^n\hat{f}(\xi)] = (-i)^n f^{(n)}(x)$.
(4′) **Multiplication by** $e^{-i\xi\alpha}$: $\mathcal{F}^{-1}[e^{-i\xi\alpha}\hat{f}(\xi)] = f(x-\alpha)$.
(5′) **Gaussians:** $\mathcal{F}^{-1}[e^{-\alpha\xi^2}] = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{x^2}{4\alpha}}$.
(6′) **Product:** $\mathcal{F}^{-1}[\hat{f}(\xi)\hat{g}(\xi)] = f(x) * g(x)$.
(7′) **Convolution:** $\mathcal{F}^{-1}[\hat{f} * \hat{g}(\xi)] = 2\pi(fg)(x)$.

Remark 24.1
It is important to mention that there exists no established convention of how to define the Fourier transform. In the literature, we can meet an equivalent definition of (24.3) with the constant $\frac{1}{\sqrt{2\pi}}$ or $\frac{1}{2\pi}$ in front of the integral. There also exist definitions with positive sign in the exponent. The reader should keep this fact in mind while working with various sources or using the transformation tables.

Practice Problems

Problem 24.1
Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 24.2
Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and initial condition

$$u_t + cu_x = 0$$

$$u(x,0) = f(x).$$

Problem 24.3
Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$u_{tt} = c^2 u_{xx},\ x \in \mathbb{R},\ t > 0$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x).$$

Problem 24.4
Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$\Delta u = u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ 0 < y < L$$

$$u(x,0) = 0$$

$$u(x,L) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Problem 24.5
Find the Fourier transform of $f(x) = e^{-|x|\alpha}$, where $\alpha > 0$.

Problem 24.6
Prove that

$$\mathcal{F}[e^{-x}H(x)] = \frac{1}{1+i\xi}$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 24.7
Prove that

$$\mathcal{F}\left[\frac{1}{1+ix}\right] = 2\pi e^{\xi}H(-\xi).$$

Problem 24.8
Prove

$$\mathcal{F}[f(x-\alpha)] = e^{-i\xi\alpha}\hat{f}(\xi).$$

Problem 24.9
Prove

$$\mathcal{F}[e^{i\alpha x}f(x)] = \hat{f}(x-\alpha).$$

Problem 24.10
Prove the following

$$\mathcal{F}[\cos(\alpha x)f(x)] = \frac{1}{2}[\hat{f}(\xi+\alpha) + \hat{f}(\xi-\alpha)]$$

$$\mathcal{F}[\sin(\alpha x)f(x)] = \frac{1}{2}[\hat{f}(\xi+\alpha) - \hat{f}(\xi-\alpha)]$$

Problem 24.11
Prove

$$\mathcal{F}[f'(x)] = (i\xi)\hat{f}(\xi).$$

Problem 24.12
Find the Fourier transform of $f(x) = 1 - |x|$ for $-1 \leq x \leq 1$ and 0 otherwise.

Problem 24.13
Find, using the definition, the Fourier transform of

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ 1 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

Problem 24.14
Find the inverse Fourier transform of $\hat{f}(\xi) = e^{-\frac{\xi^2}{2}}$.

Problem 24.15
Find $\mathcal{F}^{-1}\left(\frac{1}{a+i\xi}\right)$.