The Fourier Transform Solutions for PDEs

In the previous chapter we discussed one class of integral transform methods, the Laplace transfom. In this chapter, we consider a second fundamental class of integral transform methods, the so-called **Fourier transform**.
Fourier series are designed to solve boundary value problems on bounded intervals. The extension of Fourier methods to the entire real line leads naturally to the Fourier transform, an extremely powerful mathematical tool for the analysis of non-periodic functions. The Fourier transform is of fundamental importance in a broad range of applications, including both ordinary and partial differential equations, quantum mechanics, signal processing, control theory, and probability, to name but a few.

23 Complex Version of Fourier Series

We have seen in Section 15 that a $2L-$periodic function $f : \mathbb{R} \to \mathbb{R}$ that is piecewise smooth on $[-L, L]$ can be expanded in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

at all points of continuity of f. In the context of Fourier analysis, this is referred to as the **real form** of the Fourier series. It is often convenient to recast this series in **complex form** by means of Euler formula

$$e^{ix} = \cos x + i \sin x.$$

It follows from this formula that

$$e^{ix} + e^{-ix} = 2\cos x \text{ and } e^{ix} - e^{-ix} = 2i \sin x$$

or

$$\cos x = \tfrac{e^{ix}+e^{-ix}}{2} \text{ and } \sin x = \tfrac{e^{ix}-e^{-ix}}{2i}.$$

Hence the Fourier expansion of f can be rewritten as

$$\begin{aligned} f(x) =& \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2} \right) \right. \\ & \left. + b_n \left(\frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2i} \right) \right] \end{aligned}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \tag{23.1}$$

where $c_0 = \frac{a_0}{2}$ and for $n \in \mathbb{N}$ we have

$$c_n = \frac{a_n - ib_n}{2}$$

$$c_{-n} = \frac{a_n + ib_n}{2}.$$

It follows that if $n \in \mathbb{N}$ then

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}). \tag{23.2}$$

That is, a_n and b_n can be easily found once we have formulas for c_n. In order to find these formulas, we need to evaluate the following integral

$$\begin{aligned}
\int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{im\pi x}{L}} dx =& \int_{-L}^{L} e^{\frac{i(n-m)\pi x}{L}} dx \\
=& \frac{L}{i(n-m)\pi} \left. e^{\frac{i(n-m)\pi x}{L}} \right]_{-L}^{L} \\
=& -\frac{iL}{(n-m)\pi}[\cos[(n-m)\pi] + i\sin[(n-m)\pi] \\
& -\cos[-(n-m)\pi] - i\sin[-(n-m)\pi]] \\
=& 0
\end{aligned}$$

if $n \neq m$. If $n = m$ then

$$\int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{in\pi x}{L}} dx = 2L.$$

Now, if we multiply (23.1) by $e^{-\frac{in\pi x}{L}}$ and integrate from $-L$ to L and apply the last result we find

$$\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx = 2Lc_n$$

which yields the formula for coefficients of the complex form of the Fourier series:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx, \quad n = 0, \pm 1, \pm 2, \cdots.$$

Example 23.1
Find the complex Fourier coefficients of the function

$$f(x) = x, \quad -\pi \leq x \leq \pi$$

extended to be periodic of period 2π.

Solution.
Using integration by parts and the fact that $e^{i\pi} = e^{-i\pi} = -1$ we find

$$\begin{aligned}
c_n =& \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx \\
=& \frac{1}{2\pi}\left[\left(\frac{ix}{n}\right)e^{-inx}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{i}{n}\right)e^{-inx}dx\right] \\
=& \frac{1}{2\pi}\left[\left(\frac{i\pi}{n}\right)e^{-in\pi} + \left(\frac{i\pi}{n}\right)e^{in\pi}\right] \\
+& \frac{1}{2\pi}\left[\frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}e^{in\pi}\right] \\
=& \frac{1}{2\pi}\left[2i\frac{\pi}{n}(-1)^n\right] + \frac{1}{2\pi}(0) = \frac{(-1)^n i}{n}
\end{aligned}$$

for $n \in \mathbb{N}$ and for $n = 0$, we have

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0 \blacksquare$$

Remark 23.1
It is often the case that the complex form of the Fourier series is far simpler to calculate than the real form. One can then use (23.2) to find the real form of the Fourier series. For example, the Fourier coefficients of the real form of the previous function are given by

$$a_n = (c_n + c_{-n}) = 0 \text{ and } b_n = i(c_n - c_{-n}) = \tfrac{2}{n}(-1)^{n+1}, \quad n \in \mathbb{N}$$

Practice Problems

Problem 23.1
Find the complex Fourier coefficients of the function

$$f(x) = x, \quad -1 \le x \le 1$$

extended to be periodic of period 2.

Problem 23.2
Let

$$f(x) = \begin{cases} 0 & -\pi < x < \frac{-\pi}{2} \\ 1 & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0 & \pi < x < \pi \end{cases}$$

be $2\pi-$periodic. Find its complex series representation.

Problem 23.3
Find the complex Fourier series of the $2\pi-$periodic function $f(x) = e^{ax}$ over the interval $(-\pi, \pi)$.

Problem 23.4
Find the complex Fourier series of the $2\pi-$periodic function $f(x) = \sin x$ over the interval $(-\pi, \pi)$.

Problem 23.5
Find the complex Fourier series of the $2\pi-$periodic function defined

$$f(x) = \begin{cases} 1 & 0 < x < T \\ 0 & T < x < 2\pi \end{cases}$$

Problem 23.6
Let $f(x) = x^2, \ -\pi < x < \pi$, be $2\pi-$periodic.
(a) Calculate the complex Fourier series representation of f.
(b) Using the complex Fourier series found in (a), recover the real Fourier series representation of f.

Problem 23.7
Let $f(x) = \sin \pi x, \ -\frac{1}{2} < x < \frac{1}{2}$, be of period 1.
(a) Calculate the coefficients a_n, b_n and c_n.
(b) Find the complex Fourier series representation of f.

Problem 23.8
Let $f(x) = 2 - x, \quad -2 < x < 2$, be of period 4.
(a) Calculate the coefficients a_n, b_n and c_n.
(b) Find the complex Fourier series representation of f.

Problem 23.9
Suppose that the coefficients c_n of the complex Fourier series are given by

$$c_n = \begin{cases} \frac{2}{i\pi n} & \text{if } |n| \text{ is odd} \\ 0 & \text{if } |n| \text{ is even.} \end{cases}$$

Find $a_n, \ n = 0, 1, 2, \cdots$ and $b_n, \ n = 1, 2, \cdots$.

Problem 23.10
Recall that any complex number z can be written as $z = Re(z) + iIm(z)$ where $Re(z)$ is called the real part of z and $Im(z)$ is called the **imaginary part**. The **complex conjugate** of z is the complex number $\overline{z} = Re(z) - iIm(z)$. Using these definitions show that $a_n = 2Re(c_n)$ and $b_n = -2Im(c_n)$.

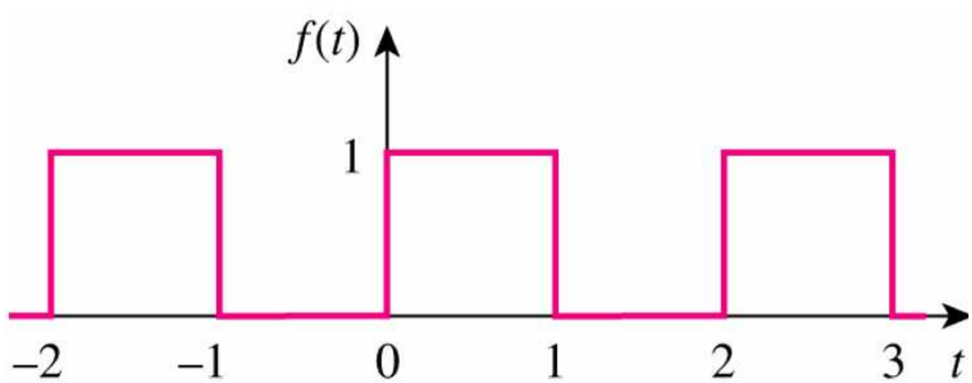
Problem 23.11
Suppose that

$$c_n = \begin{cases} \frac{i}{2\pi n}[e^{-inT} - 1] & \text{if } n \neq 0 \\ \frac{T}{2\pi} & \text{if } n = 0. \end{cases}$$

Find a_n and b_n.

Problem 23.12
Find the complex Fourier series of the function $f(x) = e^x$ on $[-2, 2]$.

Problem 23.13
Consider the wave form



(a) Write $f(x)$ explicitly. What is the period of f.
(b) Determine a_0 and a_n for $n \in \mathbb{N}$.
(c) Determine b_n for $n \in \mathbb{N}$.
(d) Determine c_0 and c_n for $n \in \mathbb{N}$.

Problem 23.14
If z is a complex number we define $\sin z = \frac{1}{2}(e^{iz} - e^{-iz})$. Find the complex form of the Fourier series for $\sin 3x$ without evaluating any integrals.