22 Solving PDEs Using Laplace Transform

The same idea for solving linear ODEs using Laplace transform can be exploited when solving PDEs for functions in two variables $u = u(x,t)$. The transformation will be done with respect to the time variable $t \geq 0$, the spatial variable x will be treated as a parameter unaffected by this transform. In particular we define the Laplace transform of $u(x,t)$ by the formula

$$\mathcal{L}(u(x,t)) = U(x,s) = \int_0^\infty u(x,t)e^{-st}dt.$$

The time derivatives are transformed in the same way as in the case of functions in one variable, that is, for example

$$\mathcal{L}(u_t)(x,t) = sU(x,s) - u(x,0)$$

and

$$\mathcal{L}(u_{tt})(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0).$$

The spatial derivatives remain unchanged, for example,

$$\mathcal{L}u_x(x,t) = \int_0^\infty u_x(x,\tau)e^{-s\tau}d\tau = \frac{\partial}{\partial x}\int_0^\infty u(x,\tau)e^{-s\tau}d\tau = U_x(x,s).$$

Likewise, we have

$$\mathcal{L}u_{xx}(x,t) = U_{xx}(x,s).$$

Thus, applying the Laplace transform to a PDE in two variables x and t we obtain an ODE in the variable x and with the parameter s.

Example 22.1

Let $u(x,t)$ be the concentration of a chemical contaminant dissolved in a liquid on a half-infinte domain $x > 0$. Let us assume that at time $t = 0$ the concentration is 0 and on the boundary $x = 0$, constant unit concentration of the contaminant is kept for $t > 0$. The behaviour of this problem is described by the following mathematical model

$$\begin{cases} u_t - u_{xx} = 0 \quad , \; x > 0, \; t > 0 \\ u(x,0) = 0, \\ u(0,t) = 1, \\ |u(x,t)| < \infty. \end{cases}$$

Find $u(x,t)$.

Solution.
Applying Laplace transform to both sides of the equation we obtain

$$sU(x,s) - u(x,0) - U_{xx}(x,s) = 0$$

or

$$U_{xx}(x,s) - sU(x,s) = 0.$$

This is a second order linear ODE in the variable x and positive parameter s. Its general solution is

$$U(x,s) = A(s)e^{-\sqrt{s}x} + B(s)e^{\sqrt{s}x}.$$

Since $U(x,s)$ is bounded in both variables, we must have $B(s) = 0$ and in this case we obtain

$$U(x,s) = A(s)e^{-\sqrt{s}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U(0,s) = \mathcal{L}(1) = \frac{1}{s}.$$

This leads to $A(s) = \frac{1}{s}$ and the transformed solution becomes

$$U(x,s) = \frac{1}{s}e^{-\sqrt{s}x}.$$

Thus,

$$u(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s}e^{-\sqrt{s}x}\right) = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} e^{-w^2}dw \blacksquare$$

Example 22.2
Solve the following initial boundary value problem

$$\begin{cases} u_t - u_{xx} = 0 \quad, \ x > 0, \ t > 0 \\ u(x,0) = 0, \\ u(0,t) = f(t), \\ |u(x,t)| < \infty. \end{cases}$$

Solution.
Following the argument of the previous example we find

$$U(x,s) = F(s)e^{-\sqrt{s}x}, \quad F(s) = \mathcal{L}f(t).$$

Thus, using Theorem 21.6 we can write

$$u(x,t) = \mathcal{L}^{-1}\left(F(s)e^{-\sqrt{s}x}\right) = f * \mathcal{L}^{-1}(e^{-\sqrt{s}x}).$$

It can be shown that

$$\mathcal{L}^{-1}(e^{-\sqrt{s}x}) = \frac{x}{\sqrt{4\pi t^3}}e^{-\frac{x^2}{4t}}.$$

Hence,

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-s)^3}}e^{-\frac{x^2}{4(t-s)}}f(s)ds \blacksquare$$

Example 22.3
Solve the wave equation

$$\begin{cases} u_{tt} - c^2u_{xx} = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = u_t(x,0) = 0, \\ u(0,t) = f(t), \\ |u(x,t)| < \infty. \end{cases}$$

Solution.
Applying Laplace transform to both sides of the equation we obtain

$$s^2U(x,s) - su(x,0) - u_t(x,0) - c^2U_{xx}(x,s) = 0$$

or

$$c^2U_{xx}(x,s) - s^2U(x,s) = 0.$$

This is a second order linear ODE in the variable x and positive parameter s. Its general solution is

$$U(x,s) = A(s)e^{-\frac{s}{c}x} + B(s)e^{\frac{s}{c}x}.$$

Since $U(x,s)$ is bounded, we must have $B(s) = 0$ and in this case we obtain

$$U(x,s) = A(s)e^{-\frac{s}{c}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U(0,s) = \mathcal{L}(f(t)) = F(s).$$

This leads to $A(s) = F(s)$ and the transformed solution becomes

$$U(x,s) = F(s)e^{-\frac{s}{c}x}.$$

Thus,

$$u(x,t) = \mathcal{L}^{-1}\left(F(s)e^{-\frac{x}{c}s}\right) = H\left(t - \frac{x}{c}\right)f\left(t - \frac{x}{c}\right) \blacksquare$$

Remark 22.1
Laplace transforms are useful in solving parabolic and some hyperbolic PDEs. They are not in general useful in solving elliptic PDEs.

Practice Problems

Problem 22.1
Solve by Laplace transform

$$\begin{cases} u_t + u_x = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = \sin x, & \\ u(0,t) = 0 & \end{cases}$$

Hint: Method of integrating factor of ODEs.

Problem 22.2
Solve by Laplace transform

$$\begin{cases} u_t + u_x = -u & , \ x > 0, \ t > 0 \\ u(x,0) = \sin x, & \\ u(0,t) = 0 & \end{cases}$$

Problem 22.3
Solve

$$u_t = 4u_{xx}$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = 2\sin \pi x + 6 \sin 2\pi x.$$

Hint: A particular solution of a second order ODE must be found using the method of variation of parameters.

Problem 22.4
Solve by Laplace transform

$$\begin{cases} u_t - u_x = u & , \ x > 0, \ t > 0 \\ u(x,0) = e^{-5x}, & \\ |u(x,t)| < \infty & \end{cases}$$

Problem 22.5
Solve by Laplace transform

$$\begin{cases} u_t + u_x = t & , \ x > 0, \ t > 0 \\ u(x,0) = 0, & \\ u(0,t) = t^2 & \end{cases}$$

Problem 22.6
Solve by Laplace transform

$$\begin{cases} xu_t + u_x = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = 0, \\ u(0,t) = t \end{cases}$$

Problem 22.7
Solve by Laplace transform

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = u_t(x,0) = 0, \\ u(0,t) = \sin t, \\ |u(x,t)| < \infty \end{cases}$$

Problem 22.8
Solve by Laplace transform

$$u_{tt} - 9u_{xx} = 0, \ 0 \le x \le \pi, \ t > 0$$

$$u(0,t) = u(\pi,t) = 0,$$

$$u_t(x,0) = 0, \quad u(x,0) = 2\sin x.$$

Problem 22.9
Solve by Laplace transform

$$\begin{cases} u_{xy} = 1 & , \ x > 0, \ y > 0 \\ u(x,0) = 1, \\ u(0,y) = y + 1. \end{cases}$$

Problem 22.10
Solve by Laplace transform

$$\begin{cases} u_{tt} = c^2 u_{xx} & , \ x > 0, \ t > 0 \\ u(x,0) = u_t(x,0) = 0, \\ u_x(0,t) = f(t), \\ |u(x,t)| < \infty. \end{cases}$$

Problem 22.11
Solve by Laplace transform

$$\begin{cases} u_t + u_x = u & , \ x > 0, \ t > 0 \\ u(x,0) = \sin x, \\ u(0,t) = 0 \end{cases}$$

Problem 22.12
Solve by Laplace transform

$$\begin{cases} u_t - c^2 u_{xx} = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = T, \\ u(0,t) = 0, \\ |u(x,t)| < \infty \end{cases}$$

Problem 22.13
Solve by Laplace transform

$$u_t - 3u_{xx} = 0, \ 0 \le x \le 2, \ t > 0$$

$$u(0,t) = u(2,t) = 0,$$
$$u(x,0) = 5\sin(\pi x)$$

Problem 22.14
Solve by Laplace transform

$$u_t - 4u_{xx} = 0, \ 0 \le x \le \pi, \ t > 0$$

$$u_x(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 40\cos\frac{x}{2}$$

Problem 22.15
Solve by Laplace transform

$$u_{tt} - 4u_{xx} = 0, \ 0 \le x \le 2, \ t > 0$$

$$u(0,t) = u(2,t) = 0,$$
$$u_t(x,0) = 0, \quad u(x,0) = 3\sin\pi x.$$