The Laplace Transform Solutions for PDEs

If in a partial differential equation the time t is one of the independent variables of the searched-for function, we say that the PDE is an **evolution equation.** Examples of evolutions equations are the heat equation and the wave equation. In contrast, when the equation involves only spatial independent variables then the equation is called a **stationary equation.** Examples of stationary equations are the Laplace's equations and Poisson equations.
There are classes of methods that can be used for solving the initial value or initial boundary problems for evolution equations. We refer to these methods as the methods of **integral transforms**. The fundamental ones are the Laplace and the Fourier transforms. In this chapter we will just consider the Laplace transform.

21 Essentials of the Laplace Transform

Laplace transform has been introduced in an ODE course, and is used especially to solve linear ODEs with constant coefficients, where the equations are transformed to algebraic equations. This idea can be easily extended to PDEs, where the transformation leads to the decrease of the number of independent variables. PDEs in two variables are thus reduced to ODEs. In this section we review the Laplace transform and its properties.
Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:
• The given "hard" problem is transformed into a "simple" equation.
• This simple equation is solved by purely algebraic manipulations.
• The solution of the simple equation is transformed back to obtain the solution of the given problem.
In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.
The above procedure can be summarized by Figure 21.1

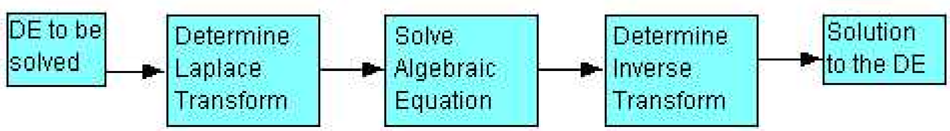


Figure 21.1

In this section we introduce the concept of Laplace transform and discuss some of its properties.
The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of f, which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T\to\infty} \int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

The integral which defines a Laplace transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. When the improper integral is convergent then we say that the function $f(t)$ possesses a Laplace transform. So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral.

Example 21.1
Find the Laplace transform, if it exists, of each of the following functions

$$(a)\ f(t) = e^{at} \quad (b)\ f(t) = 1 \quad (c)\ f(t) = t \quad (d)\ f(t) = e^{t^2}$$

Solution.
(a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t} dt = \lim_{T\to\infty} \int_0^T e^{-(s-a)t} dt.$$

But

$$\int_0^T e^{-(s-a)t} dt = \begin{cases} T & \text{if } s = a \\ \frac{1-e^{-(s-a)T}}{s-a} & \text{if } s \neq a. \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, \quad s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^\infty e^{-st} dt = \lim_{T\to\infty} \int_0^T e^{-st} dt = \frac{1}{s}, \ s > 0.$$

(c) We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st} dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^\infty = \frac{1}{s^2}, \ s > 0.$$

(d) Again using the definition of Laplace transform we find

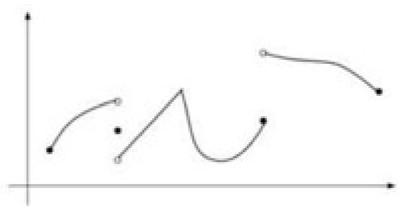
$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2 - st} dt.$$

If $s \leq 0$ then $t^2 - st \geq 0$ so that $e^{t^2-st} \geq 1$ and this implies that $\int_0^\infty e^{t^2-st} dt \geq \int_0^\infty dt$. Since the integral on the right is divergent, by the comparison theorem of improper integrals, the integral on the left is also divergent. Now, if $s > 0$ then $\int_0^\infty e^{t(t-s)} dt \geq \int_s^\infty dt$. By the same reasoning the integral on the left is divergent. This shows that the function $f(t) = e^{t^2}$ does not possess a Laplace transform ■

The above example raises the question of what class or classes of functions possess a Laplace transform. To answer this question we introduce few mathematical concepts.
A function f that satisfies

$$|f(t)| \leq Me^{at}, \quad t \geq C \tag{21.1}$$

is said to be a function with an **exponential order at infinity**. A function f is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e. the subinterval without its endpoints) and has a finite limit at the endpoints (**jump discontinuities** and no vertical asymptotes) of each subinterval. Below is a sketch of a piecewise continuous function.



Note that a piecewise continuous function is a function that has a finite number of breaks in it and doesn't blow up to infinity anywhere. A function defined for $t \geq 0$ is said to be **piecewise continuous on the infinite interval** if it is piecewise continuous on $0 \leq t \leq T$ for all $T > 0$.

Example 21.2
Show that the following functions are piecewise continuous and of exponential order at infinity for $t \geq 0$

$$\text{(a) } f(t) = t^n \quad \text{(b) } f(t) = t^n \sin at$$

Solution.
(a) Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$, $t^n \leq n!e^t$. Hence, t^n is piecewise continuous and of exponential order at infinity.
(b) Since $|t^n \sin at| \leq n!e^t$, $t^n \sin at$ is piecewise continuous and of exponential order at infinity ■

The following is an existence result of Laplace transform.

Theorem 21.1
Suppose that $f(t)$ is piecewise continuous on $t \geq 0$ and has an exponential order at infinity with $|f(t)| \leq Me^{at}$ for $t \geq C$. Then the Laplace transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

exists as long as $s > a$. Note that the two conditions above are sufficient, but not necessary, for $F(s)$ to exist.

In what follows, we will denote the class of all piecewise continuous functions with exponential order at infinity by $\mathcal{PE}$. The next theorem shows that any linear combination of functions in $\mathcal{PE}$ is also in $\mathcal{PE}$. The same is true for the product of two functions in $\mathcal{PE}$.

Theorem 21.2
Suppose that $f(t)$ and $g(t)$ are two elements of $\mathcal{PE}$ with

$$|f(t)| \leq M_1e^{a_1t}, \quad t \geq C_1 \quad \text{and} \quad |g(t)| \leq M_2e^{a_1t}, \quad t \geq C_2.$$

(i) For any constants α and β the function $\alpha f(t) + \beta g(t)$ is also a member of $\mathcal{PE}$. Moreover

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)].$$

(ii) The function $h(t) = f(t)g(t)$ is an element of $\mathcal{PE}$.

We next discuss the problem of how to determine the function $f(t)$ if $F(s)$ is given. That is, how do we invert the transform. The following result on uniqueness provides a possible answer. This result establishes a one-to-one correspondence between the set $\mathcal{PE}$ and its Laplace transforms. Alternatively, the following theorem asserts that the Laplace transform of a member in $\mathcal{PE}$ is unique.

Theorem 21.3
Let $f(t)$ and $g(t)$ be two elements in $\mathcal{PE}$ with Laplace transforms $F(s)$ and $G(s)$ such that $F(s) = G(s)$ for some $s > a$. Then $f(t) = g(t)$ for all $t \geq 0$ where both functions are continuous.

With the above theorem, we can now officially define the inverse Laplace transform as follows: For a piecewise continuous function f of exponential order at infinity whose Laplace transform is F, we call f the **inverse Laplace transform** of F and write $f = \mathcal{L}^{-1}[F(s)]$. Symbolically

$$f(t) = \mathcal{L}^{-1}[F(s)] \Longleftrightarrow F(s) = \mathcal{L}[f(t)].$$

Example 21.3
Find $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right), \quad s > 1$.

Solution.
From Example 21.1(a), we have that $\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a$. In particular, for $a = 1$ we find that $\mathcal{L}[e^t] = \frac{1}{s-1}, \ s > 1$. Hence, $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t, \ t \geq 0$ ■.

The above theorem states that if $f(t)$ is continuous and has a Laplace transform $F(s)$, then there is no other function that has the same Laplace transform. To find $\mathcal{L}^{-1}[F(s)]$, we can inspect tables of Laplace transforms of known functions to find a particular $f(t)$ that yields the given $F(s)$.
When the function $f(t)$ is not continuous, the uniqueness of the inverse Laplace transform is not assured. The following example addresses the uniqueness issue.

Example 21.4
Consider the two functions $f(t) = H(t)H(3-t)$ and $g(t) = H(t) - H(t-3)$, where H is the Heaviside function defined by

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

(a) Are the two functions identical?
(b) Show that $\mathcal{L}[f(t)] = \mathcal{L}[g(t)$.

Solution.
(a) We have

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 3 \\ 0, & t > 3 \end{cases}$$

and

$$g(t) = \begin{cases} 1, & 0 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

Since $f(3) = 1$ and $g(3) = 0$, f and g are not identical.
(b) We have

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = \int_0^3 e^{-st}dt = \frac{1-e^{-3s}}{s}, s > 0.$$

Thus, both functions $f(t)$ and $g(t)$ have the same Laplace transform even though they are not identical. However, they are equal on the interval(s) where they are both continuous ■

The inverse Laplace transform possesses a linear property as indicated in the following result.

Theorem 21.4
Given two Laplace transforms $F(s)$ and $G(s)$ then

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

for any constants a and b.

Convolution integrals are useful when finding the inverse Laplace transform of products. They are defined as follows: The **convolution** of two scalar piecewise continuous functions $f(t)$ and $g(t)$ defined for $t \geq 0$ is the integral

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

Example 21.5
Find $f * g$ where $f(t) = e^{-t}$ and $g(t) = \sin t$.

Solution.
Using integration by parts twice we arrive at

$$\begin{aligned}(f * g)(t) =& \int_0^t e^{-(t-s)} \sin s ds \\ =& \frac{1}{2}\left[e^{-(t-s)}(\sin s - \cos s)\right]_0^t \\ =& \frac{e^{-t}}{2} + \frac{1}{2}(\sin t - \cos t) \ ■ \end{aligned} \tag{21.2}$$

Next, we state several properties of convolution product, which resemble those of ordinary product.

Theorem 21.5
Let $f(t), g(t)$, and $k(t)$ be three piecewise continuous scalar functions defined for $t \geq 0$ and c_1 and c_2 are arbitrary constants. Then
(i) $f * g = g * f$ (Commutative Law)
(ii) $(f * g) * k = f * (g * k)$ (Associative Law)
(iii) $f * (c_1 g + c_2 k) = c_1 f * g + c_2 f * k$ (Distributive Law)

Example 21.6
Express the solution to the initial value problem $y' + \alpha y = g(t),\ y(0) = y_0$ in terms of a convolution integral.

Solution.
Solving this initial value problem by the method of integrating factor we find

$$y(t) = e^{-\alpha t} y_0 + \int_0^t e^{-\alpha(t-s)} g(s) ds = e^{-\alpha t} y_0 + (e^{-\alpha t} * g)(t) \blacksquare$$

The following theorem, known as the Convolution Theorem, provides a way for finding the Laplace transform of a convolution integral and also finding the inverse Laplace transform of a product.

Theorem 21.6
If $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$, and of exponential order at infinity then

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)] \mathcal{L}[g(t)] = F(s)G(s).$$

Thus, $(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)]$.

Example 21.7
Use the convolution theorem to find the inverse Laplace transform of

$$P(s) = \frac{1}{(s^2 + a^2)^2}.$$

Solution.
Note that

$$P(s) = \left(\frac{1}{s^2 + a^2}\right) \left(\frac{1}{s^2 + a^2}\right).$$

So, in this case we have, $F(s) = G(s) = \frac{1}{s^2+a^2}$ so that $f(t) = g(t) = \frac{1}{a} \sin(at)$. Thus,

$$(f * g)(t) = \frac{1}{a^2} \int_0^t \sin(at - as) \sin(as) ds = \frac{1}{2a^3} (\sin(at) - at \cos(at)) \blacksquare$$

Example 21.8
Solve the initial value problem

$$4y'' + y = g(t), \quad y(0) = 3, \; y'(0) = -7$$

Solution.
Take the Laplace transform of all the terms and plug in the initial conditions to obtain

$$4(s^2Y(s) - 3s + 7) + Y(s) = G(s)$$

or

$$(4s^2 + 1)Y(s) - 12s + 28 = G(s).$$

Solving for $Y(s)$ we find

$$\begin{aligned} Y(s) =& \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)} \\ =& \frac{3s}{s^2 + \left((\frac{1}{2}\right)^2} - 7\frac{\left(\frac{1}{2}\right)^2}{s^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{4}G(s)\frac{\left(\frac{1}{2}\right)^2}{s^2 + \left(\frac{1}{2}\right)^2} \end{aligned}$$

Hence,

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 7\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{s}{2}\right)g(t-s)ds.$$

So, once we decide on a $g(t)$ all we need to do is to evaluate the integral and we'll have the solution ■

We conclude this section with the following table of Laplace transform pairs where H is the Heaviside function defined by $H(t) = 1$ for $t \geq 0$ and 0 otherwise.

f(t)	F(s)
$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{s},\ s > 0$
$t^n,\ n = 1, 2, \cdots$	$\frac{n!}{s^{n+1}},\ s > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha},\ s > \alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ s > 0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2},\ s > \lvert\omega\rvert$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2},\ s > \lvert\omega\rvert$
$e^{\alpha t} f(t),\ with\ \lvert f(t)\rvert \leq Me^{at}$	$F(s-\alpha),\ s > \alpha + a$
$e^{\alpha t} H(t)$	$\frac{1}{s-\alpha},\ s > \alpha$
$e^{\alpha t} t^n,\ n = 1, 2, \cdots$	$\frac{n!}{(s-\alpha)^{n+1}},\ s > \alpha$
$e^{\alpha t} \sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2},\ s > \alpha$
$e^{\alpha t} \cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2},\ s > \alpha$
$f(t-\alpha)H(t-\alpha),\ \alpha \geq 0$ $with\ \lvert f(t)\rvert \leq Me^{at}$	$e^{-\alpha s} F(s),\ s > a$
$H(t-\alpha),\ \alpha \geq 0$	$\frac{e^{-\alpha s}}{s},\ s > 0$
$tf(t)$	-$F'(s)$
$\frac{t}{2\omega} \sin \omega t$	$\frac{s}{(s^2+\omega^2)^2},\ s > 0$
$\frac{1}{2\omega^3}[\sin \omega t - \omega t \cos \omega t]$	$\frac{1}{(s^2+\omega^2)^2},\ s > 0$
$f'(t),\ with\ f(t)\ continuous$ $and\ \lvert f'(t)\rvert \leq Me^{at}$	$sF(s) - f(0)$ $s > \max\{a, 0\} + 1$
$f''(t),\ with\ f'(t)\ continuous$ $and\ \lvert f''(t)\rvert \leq Me^{at}$	$s^2 F(s) - sf(0) - f'(0)$ $s > \max\{a, 0\} + 1$
$f^{(n)}(t),\ with\ f^{(n-1)}(t)\ continuous$ $and\ \lvert f^{(n)}(t)\rvert \leq Me^{at}$	$s^n F(s) - s^{n-1} f(0) - \cdots$ -$sf^{(n-2)}(0) - f^{(n-1)}(0)$ $s > \max\{a, 0\} + 1$
$\frac{2}{2\sqrt{\pi}} \int_{\frac{\alpha}{2\sqrt{t}}}^{\infty} e^{-u^2} du$	$\frac{e^{-\alpha\sqrt{s}}}{s}$
$\int_0^t f(u)du,\ with\ \lvert f(t)\rvert \leq Me^{at}$	$\frac{F(s)}{s},\ \ s > \max\{a, 0\} + 1$

Table $\mathcal{L}$

Practice Problems

Problem 21.1
Determine whether the integral $\int_0^\infty \frac{1}{1+t^2}dt$ converges. If the integral converges, give its value.

Problem 21.2
Determine whether the integral $\int_0^\infty \frac{t}{1+t^2}dt$ converges. If the integral converges, give its value.

Problem 21.3
Determine whether the integral $\int_0^\infty e^{-t}\cos{(e^{-t})}dt$ converges. If the integral converges, give its value.

Problem 21.4
Using the definition, find $\mathcal{L}[e^{3t}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.5
Using the definition, find $\mathcal{L}[t-5]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.6
Using the definition, find $\mathcal{L}[e^{(t-1)^2}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.7
Using the definition, find $\mathcal{L}[(t-2)^2]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.8
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & t \ge 1 \end{cases}$$

Problem 21.9
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t-1, & 1 \leq t < 2 \\ 0, & t \geq 2. \end{cases}$$

Problem 21.10
Let n be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st} dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s} \int t^{n-1} e^{-st} dt, \quad s > 0.$$

Problem 21.11
For $s > 0$ and n a positive integer evaluate the limits

(a) $\lim_{t \to 0} t^n e^{-st}$ (b) $\lim_{t \to \infty} t^n e^{-st}$

Problem 21.12
Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$. Find the domain of $F(s)$.

Problem 21.13
Find $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)$.

Problem 21.14
Find $\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right)$.

Problem 21.15
Find $\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right)$.

Problem 21.16
Use Table $\mathcal{L}$ to find $\mathcal{L}[2e^t + 5]$.

Problem 21.17
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{3t-3}H(t-1)]$.

Problem 21.18
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin^2 \omega t]$.

Problem 21.19
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin 3t \cos 3t]$.

Problem 21.20
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{2t} \cos 3t]$.

Problem 21.21
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{4t}(t^2 + 3t + 5)]$.

Problem 21.22
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{10}{s^2+25} + \frac{4}{s-3}]$.

Problem 21.23
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$.

Problem 21.24
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$.

Problem 21.25
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right]$.

Problem 21.26
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{s^2-9}\right]$.

Problem 21.27
Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 12, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

Problem 21.28
Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 21.29
Consider the functions $f(t) = e^t$ and $g(t) = e^{-2t},\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Problem 21.30
Consider the functions $f(t) = \sin t$ and $g(t) = \cos t,\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Problem 21.31
Compute $t * t * t$.

Problem 21.32
Compute $H(t) * e^{-t} * e^{-2t}$.

Problem 21.33
Compute $t * e^{-t} * e^t$.