20 Laplace's Equations in Circular Regions

In the previous section we solved the Dirichlet problem for Laplace's equation on a rectangular region. However, if the domain of the solution is a disc, an annulus, or a circular wedge, it is useful to study the two-dimensional Laplace's equation in polar coordinates.
It is well known in calculus that the cartesian coordinates (x, y) and the polar coordinates (r, θ) of a point are related by the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ and $\tan\theta = \frac{y}{x}$. Using the chain rule we obtain

$$\begin{aligned}
u_x =& u_r r_x + u_\theta \theta_x = \cos\theta u_r - \frac{\sin\theta}{r} u_\theta \\
u_{xx} =& u_{xr} r_x + u_{x\theta}\theta_x \\
=& \left(\cos\theta u_{rr} + \frac{\sin\theta}{r^2} u_\theta - \frac{\sin\theta}{r} u_{r\theta}\right)\cos\theta \\
&+ \left(-\sin\theta u_r + \cos\theta u_{r\theta} - \frac{\cos\theta}{r} u_\theta - \frac{\sin\theta}{r} u_{\theta\theta}\right)\left(-\frac{\sin\theta}{r}\right) \\
u_y =& u_r r_y + u_\theta \theta_y = \sin\theta u_r + \frac{\cos\theta}{r} u_\theta \\
u_{yy} =& u_{yr} r_y + u_{y\theta}\theta_y \\
=& \left(\sin\theta u_{rr} - \frac{\cos\theta}{r^2} u_\theta + \frac{\cos\theta}{r} u_{r\theta}\right)\sin\theta \\
&+ \left(\cos\theta u_r + \sin\theta u_{r\theta} - \frac{\sin\theta}{r} u_\theta + \frac{\cos\theta}{r} u_{\theta\theta}\right)\left(\frac{\cos\theta}{r}\right).
\end{aligned}$$

Substituting these equations into $\Delta u = 0$ we obtain

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{20.1}$$

Example 20.1
Find the solution to

$$\Delta u = 0, \quad x^2 + y^2 < a^2$$

subject to
(i) Boundary condition: $u(a, \theta) = f(\theta), \quad 0 \le \theta \le 2\pi$.
(ii) Boundedness at the origin: $|u(0, \theta)| < \infty$.
(iii) Periodicity: $u(r, \theta + 2\pi) = u(r, \theta),\ 0 \le \theta \le 2\pi$.

Solution.
First, note that (iii) implies that $u(r,0) = u(r,2\pi)$ and $u_\theta(r,0) = u_\theta(r,2\pi)$. Next, we will apply the method of separation of variables to (20.1). Suppose that a solution $u(r,\theta)$ of (20.1) can be written in the form $u(r,\theta) = R(r)\Theta(\theta)$. Substituting in (20.1) we obtain

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0.$$

Dividing by $R\Theta$ (under the assumption that $R\Theta \neq 0$) we obtain

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -r^2\frac{R''(r)}{R(r)} - r\frac{R'(r)}{R(r)}.$$

The left-hand side is independent of r whereas the right-hand side is independent of θ so that there is a constant λ such that

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = \lambda.$$

This results in the following ODEs

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \tag{20.2}$$

and

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0. \tag{20.3}$$

The second equation is known as **Euler's equation.** Both of these equations are easily solvable. To solve (20.2), We only have to add the appropriate boundary conditions. We have $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. The periodicity of Θ implies that $\lambda = n^2$ and Θ must be of the form

$$\Theta_n(\theta) = A'_n \cos n\theta + B'_n \sin n\theta, n = 0, 1, 2 \cdots$$

The equation in R is of Euler type and its solution must be of the form $R(r) = r^\alpha$. Since $\lambda = n^2$, the corresponding characteristic equation is

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2r^\alpha = 0.$$

Solving this equation we find $\alpha = \pm n$. Hence, we let

$$R_n(r) = C_nr^n + D_nr^{-n}, n \in \mathbb{N}.$$

For $n = 0$, $R = 1$ is a solution. To find a second solution, we solve the equation

$$r^2 R'' + rR' = 0.$$

This can be done by dividing through by r and using the substitution $S = R'$ to obtain $rS' + S = 0$. Solving this by noting that the left-hand side is just $(rS)'$ we find $S = \frac{c}{r}$. Hence, $R' = \frac{c}{r}$ and this implies $R(r) = C \ln r$. Thus, $R = 1$ and $R = \ln r$ form a fundamental set of solutions of (20.3) and so a general solution is given by

$$R_0(r) = C_0 + D_0 \ln r.$$

By assumption (ii), $u(r, \theta)$ must be bounded at $r = 0$, and so does R_n. Since r^{-n} and $\ln r$ are unbounded at $r = 0$, we must set $D_0 = D_n = 0$. In this case, the solutions to Euler's equation are given by

$$R_n(r) = C_n r^n, \ n = 0, 1, 2, \cdots .$$

Using the superposition principle, and combining the results obtained above, we find

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

Now, using the boundary condition $u(a, \theta) = f(\theta)$ we can write

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} (a^n A_n \cos n\theta + a^n B_n \sin n\theta)$$

which is usually written in a more convenient equivalent form by

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

It is obvious that a_n and b_n are the Fourier coefficients, and therefore can be determined by the formulas

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta, \quad n = 0, 1, \cdots$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta, \quad n = 1, 2, \cdots .$$

Finally, the general solution to our problem is given by

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

where

$$\begin{aligned} C_0 =& \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \\ A_n =& \frac{a_n}{a^n} = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta, \quad n = 1, 2, \cdots \\ B_n =& \frac{b_n}{a^n} = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta, \quad n = 1, 2, \cdots \blacksquare \end{aligned}$$

Example 20.2
Solve

$$\Delta u = 0,\ 0 \le \theta < 2\pi,\ 1 \le r \le 2$$

subject to

$$u(1,\theta) = u(2,\theta) = \sin\theta,\ 0 \le \theta < 2\pi.$$

Solution.
Use separation of variables. First, solving for $\Theta(\theta)$, we see that in order to ensure that the solution is $2\pi-$periodic in θ, the eigenvalues are $\lambda = n^2$. When solving the equation for $R(r)$, we do NOT need to throw out solutions which are not bounded as $r \to 0$. This is because we are working in the annulus where r is bounded away from 0 and ∞. Therefore, we obtain the general solution

$$u(r,\theta) = (C_0 + C_1 \ln r) + \sum_{n=1}^{\infty}[(C_n r^n + D_n r^{-n})\cos n\theta + (A_n r^n + B_n r^{-n})\sin n\theta].$$

But

$$C_0 + \sum_{n=1}^{\infty}[(C_n + D_n)\cos n\theta + (A_n + B_n)\sin n\theta] = \sin\theta$$

and

$$C_0 + \sum_{n=1}^{\infty}[(C_n 2^n + D_n 2^{-n})\cos n\theta + (A_n 2^n + B_n 2^{-n})\sin n\theta] = \sin\theta$$

Hence, comparing coefficients we must have

$$\begin{aligned} C_0 =& 0\\ C_n + D_n =& 0\\ A_n + B_n =& 0 \quad (n \neq 1)\\ A_1 + B_1 =& 1\\ C_n 2^n + D_n 2^{-n} =& 0\\ A_n 2^n + B_n 2^{-n} =& 0 \quad (n \neq 1)\\ 2A_1 + 2^{-1} B_1 =& 1. \end{aligned}$$

Solving these equations we find $C_0 = C_n = D_n = 0, A_1 = \frac{1}{3}, B_1 = \frac{2}{3}$, and $A_n = B_n = 0$ for $n \neq 1$. Hence, the solution to the problem is

$$u(r, \theta) = \frac{1}{3}\left(r + \frac{2}{r}\right)\sin\theta \blacksquare$$

Example 20.3
Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions

$$u_\theta(r, \theta) = 0, \;\; u_\theta(r, \frac{\pi}{3}) = 0, \;\; u(a, \theta) = \frac{1}{3}\cos 9\theta - \frac{1}{9}\cos 3\theta.$$

You may assume that the solution remains bounded as $r \to 0$.

Solution.
Separating the variables we obtain the eigenvalue problem

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

$$\Theta'(0) = \Theta'\left(\frac{\pi}{3}\right) = 0.$$

As above, because of periodicity we expect the solution to be of the form

$$\Theta(\theta) = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta.$$

The condition $\Theta'(0) = 0$ implies $B = 0$. The condition $\Theta'\left(\frac{\pi}{3}\right) = 0$ implies $\lambda_n = (3n)^2, \;\; n = 0, 1, 2, \cdots$. Thus, the angular solution is

$$\Theta_n(\theta) = A_n \cos 3n\theta, \quad n = 0, 1, 2, \cdots$$

The corresponding solutions of the radial problem are

$$R_n(r) = B_n r^{3n} + C_n r^{-3n}, \; n = 0, 1, \cdots .$$

To obtain a solution that remains bounded as $r \to 0$ we take $C_n = 0$. Hence,

$$u(r,\theta) = \sum_{n=0}^{\infty} D_n r^{3n} \cos 3n\theta, \quad n = 0, 1, 2, \cdots$$

Using the boundary condition

$$u(a,\theta) = \frac{1}{3}\cos 9\theta - \frac{1}{9}\cos 3\theta$$

we obtain $D_1 a^3 = -\frac{1}{9}$ and $D_3 a^9 = \frac{1}{3}$ and 0 otherwise. Thus,

$$u(r,\theta) = \frac{1}{3}\left(\frac{r}{a}\right)^9 \cos 9\theta - \frac{1}{9}\left(\frac{r}{a}\right)^3 \cos 3\theta \blacksquare$$

Practice Problems

Problem 20.1
Solve the Laplace's equation as in Example 20.1 in the unit disk with $u(1,\theta) = 3\sin 5\theta$.

Problem 20.2
Solve the Laplace's equation in the upper half of the unit disk with $u(1,\theta) = \pi - \theta$.

Problem 20.3
Solve the Laplace's equation in the unit disk with $u_r(1,\theta) = 2\cos 2\theta$.

Problem 20.4
Consider

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

with

$$\begin{aligned} C_0 =& \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi \\ A_n =& \frac{a_n}{a^n} = \frac{1}{a^n\pi}\int_0^{2\pi} f(\phi)\cos n\phi d\phi, \quad n = 1, 2, \cdots \\ B_n =& \frac{b_n}{a^n} = \frac{1}{a^n\pi}\int_0^{2\pi} f(\phi)\sin n\phi d\phi, \quad n = 1, 2, \cdots \end{aligned}$$

Using the trigonometric identity

$$\cos a \cos b + \sin a \sin b = \cos(a - b)$$

show that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \phi)\right] d\phi.$$

Problem 20.5
(a) Using Euler's formula from complex analysis $e^{it} = \cos t + i\sin t$ show that

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}),$$

where $i = \sqrt{-1}$.
(b) Show that

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi) = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}.$$

(c) Let $q_1 = \frac{r}{a}e^{i(\theta-\phi)}$ and $q_2 = \frac{r}{a}e^{-i(\theta-\phi)}$. It is defined in complex analysis that the absolute value of a complex number $z = x+iy$ is given by $|z| = (x^2+y^2)^{\frac{1}{2}}$. Using these concepts, show that $|q_1| < 1$ and $|q_2| < 1$.

Problem 20.6
(a)Show that

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} = \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} = \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}}$$

Hint: Each sum is a geoemtric series with a ratio less than 1 in absolute value so that these series converges.
(b) Show that

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi) = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}.$$

Problem 20.7
Show that

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi} \frac{f(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}d\phi.$$

This is known as the **Poisson formula** in polar coordinates.

Problem 20.8
Solve

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1$$

subject to

$$u(1,\theta) = \theta, \quad -\pi \le \theta \le \pi.$$

Problem 20.9
The vibrations of a symmetric circular membrane where the displacement $u(r,t)$ depends on r and t only can be describe by the one-dimensional wave equation in polar coordinates

$$u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r), \quad 0 < r < a, \ t > 0$$

with initial condition

$$u(a,t) = 0, \quad t > 0$$

and boundary conditions

$$u(r,0) = f(r), \quad u_t(r,0) = g(r), \quad 0 < r < a.$$

(a) Show that the assumption $u(r,t) = R(r)T(t)$ leads to the equation

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{R}R'' + \frac{1}{r}\frac{R'}{R} = \lambda.$$

(b) Show that $\lambda < 0$.

Problem 20.10
Cartesian coordinates and cylindrical coordinates are shown in Figure 20.1 below.

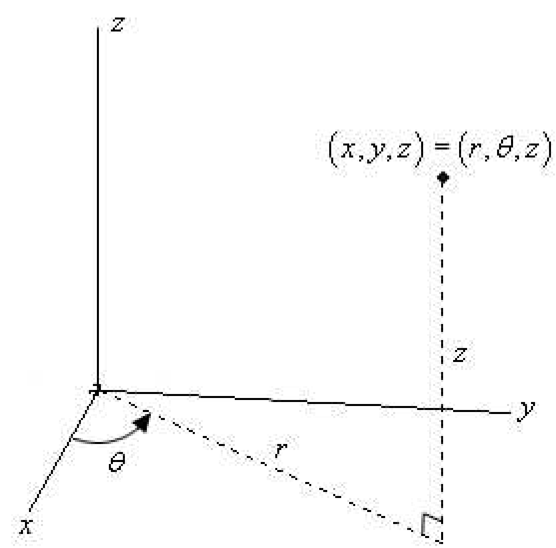

Figure 20.1

(a) Show that $x = r\cos\theta,\ y = r\sin\theta,\quad z = z$.
(b) Show that

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}.$$

Problem 20.11
An important result about harmonic functions is the so-called the **maximum principle** which states: Any harmonic function $u(x, y)$ defined in a domain Ω satisfies the inequality

$$\min_{(x,y)\in\partial\Omega} u(x, y) \le u(x, y) \le \max_{(x,y)\in\partial\Omega} u(x, y), \quad \forall(x, y) \in \Omega \cup \partial\Omega$$

where $\partial\Omega$ denotes the boundary of Ω.
Let u be harmonic in $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and satisfies $u(x, y) = 2 - x$ for all $(x, y) \in \partial\Omega$. Show that $u(x, y) > 0$ for all $(x, y) \in \Omega$.

Problem 20.12
Let u be harmonic in $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and satisfies $u(x, y) = 1 + 3x$ for all $(x, y) \in \partial\Omega$. Determine
(i) $\max_{(x,y)\in\Omega} u(x, y)$
(ii) $\min_{(x,y)\in\Omega} u(x, y)$
without solving $\Delta u = 0$.

Problem 20.13
Let $u_1(x, y)$ and $u_2(x, y)$ be harmonic functions on a smooth domain Ω such that

$$u_1|_{\partial\Omega} = g_1(x, y) \text{ and } u_2|_{\partial\Omega} = g_3(x, y)$$

where g_1 and g_2 are continuous functions satisfying

$$\max_{(x,y)\in\partial\Omega} g_1(x, y) < \min_{(x,y)\in\partial\Omega} g_1(x, y).$$

Prove that $u_1(x, y) < u_2(x, y)$ for all $(x, y) \in \Omega \cup \partial\Omega$.

Problem 20.14
Show that $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ satisfy Laplace's equation in polar coordinates.

Problem 20.15
Solve the Dirichlet problem

$$\Delta u = 0, \quad 0 \leq r < a, \quad -\pi \leq \theta \leq \pi$$

$$u(a, \theta) = \sin^2 \theta.$$

Problem 20.16
Solve Laplace's equation

$$u_{xx} + u_{yy} = 0$$

outside a circular disk ($r \geq a$) subject to the boundary condition

$$u(a, \theta) = \ln 2 + 4 \cos 3\theta.$$

You may assume that the solution remains bounded as $r \to \infty$.