2 Solutions to PDEs

By a **classical solution** or **strong solution** to a partial differential equation we mean a smooth function (i.e. the function and its derivatives are continuous up to a certain order on a common domain) that satisfies the equation. That is, by plugging in the function and its derivatives in the differential equation we obtain a true identity. A PDE might have many classical solutions. To **solve** a PDE is to find all its classical solutions. In the case of only two independent variables x and y, a classical solution $u(x,y)$ is visualized geometrically as a surface, called a **solution surface** or an **integral surface**[1] of the PDE in the (x,y,u) space.
Solving a partial differential equation is finding all the possible solutions.

Example 2.1
Show that $u(x,t) = e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x)$ is a solution to the equation $u_t - \alpha^2 u_{xx} = 0$.

Solution.
Since

$$\begin{aligned} u_t - \alpha^2 u_{xx} =& -\lambda^2\alpha^2 e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x) \\ & -\alpha^2 e^{-\lambda^2\alpha^2 t}(-\lambda^2\cos\lambda x + \lambda^2\sin\lambda x) = 0, \end{aligned}$$

the given function is a classical solution to the given equation ■

Example 2.2
The function $u(x,y) = x^2 - y^2$ is a solution to Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

Represent this solution graphically.

Solution.
The given integral surface is the hyperbolic paraboloid shown in Figure 2.1.

[1]The idea behind the name is due to the fact that integration is being used to finding the solution.

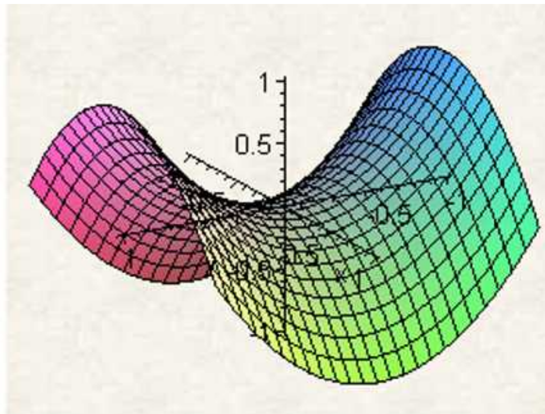

Figure 2.1

Example 2.3
Find the general solution of $u_{xy} = 0$.

Solution.
Integrating first we respect to y we find $u_x(x, y) = f(x)$, where f is an arbitrary differentiable function. Integrating u_x with respect to x we find $u(x, y) = \int f(x)dx + g(y)$, where g is an arbitrary differentiable function ■

Note that $u(x, y) = \int f(x)dx + g(y)$ in the previous example represents a family of classical solutions to the given PDE. Such an expression involves two arbitrary functions. This is in contrast to the family of solutions of an ordinary differential equation which involves arbitrary constants.
Usually, a classical solution enjoys properties such as **smootheness** (i.e. differentiability) and continuity. However, in the theory of non-linear pdes, there are solutions that do not require the smoothness property. Such solutions are called **weak solutions** or **generalized solutions**. For example, $u(x, y) = x$ is a classical solution to the differential equation $uu_x = x$. In contrast, $u(x, y) = |x|$ is a generalized solution since it is not differentiable at (0,0). In this book, the word solution will refer to a classical solution.

Example 2.4
Show that $u(x, t) = t + \frac{1}{2}x^2$ is a classical solution to the PDE

$$u_t = u_{xx}. \tag{2.1}$$

Solution.
Assume that the domain of definition of u is $D \subset \mathbb{R}^2$. Since $u, u_t, u_x, u_{tx}, u_{xx}$ exist and are continuous in D(i.e., u is smooth in D) and u satisfies equation (2.1), we conclude that u is a classical solution to the given PDE ■

We next consider the structure of solutions to linear partial differential equations. To this end, consider the linear differential operator L as defined in the previous section. The defining properties of linearity immediately imply the key facts concerning homogeneous linear differential equations.

Theorem 2.1
The sum of two solutions to a homogeneous linear differential equation is again a solution, as is the product of a solution by any constant.

Proof.
Let u_1, u_2 be solutions, meaning that $L[u_1] = 0$ and $L[u_2] = 0$. Then, thanks to linearity,

$$L[u_1 + u_2] = L[u_1] + L[u_2] = 0,$$

and hence their sum $u_1 + u_2$ is a solution. Similarly, if α is any constant, and u is any solution, then

$$L[\alpha u] = \alpha L[u] = \alpha 0 = 0,$$

and so the scalar multiple αu is also a solution ■

The following result is known as the **superposition principle** for homogeneous linear equations. It states that from given solutions to the equation one can create many more solutions.

Theorem 2.2
If $u_1, \cdots, u_n$ are solutions to a common homogeneous linear partial differential equation $L[u] = 0$, then the linear combination $u = c_1u_1 + \cdots + c_nu_n$ is a solution for any choice of constants $c_1, \cdots, c_n$.

Proof.
The key fact is that, thanks to the linearity of L, for any sufficiently smooth functions $u_1, \cdots, u_n$ and any constants $c_1, \cdots, c_n$,

$$\begin{aligned} L[u] =& L[c_1u_1 + \cdots + c_nu_n] = L[c_1u_1 + \cdots + c_{n-1}u_{n-1}] + L[c_nu_n] \\ =& \cdots = L[c_1u_1] + \cdots + L[c_nu_n] = c_1L[u_1] + \cdots + c_nL[u_n]. \end{aligned}$$

Since $L[u_1] = 0, \cdots, L[u_n] = 0$, then the right hand side of the above equation vanishes, proving that u is also a solution to the homogeneous equation $L[u] = 0$ ■

As you have noticed by the above discussion, one or more solutions of a linear homogeneous PDE leads to the creation of lots of solutions according to the Principle of Superposition. In contrast, the Principle of Superposition does not apply to non-homogeneous linear PDEs as shown in the next example.

Example 2.5
Consider the differential equation $u_x = 1$.
(a) Show that the functions $u_1(x,y) = x$ and $u_2(x,y) = x + 1$ are solutions to the given differentail equation.
(b) Show that the function $u(x,y) = u_1(x,y) + u_2(x,y) = 2x + 1$ is not a solution.

Solution.
(a) By simple differentiation we find $\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 1$.
(b) Since $\frac{\partial u}{\partial x} = 2 \neq 1$, the function $u(x,y)$ is not a solution ■

In physical applications, homogeneous linear equations model unforced systems that are subject to their own internal constraints. External forcing is represented by an additional term that does not involve the dependent variable. This results in the non-homogeneous equation

$$L[u] = f$$

where L is a linear partial differential operator, u is the dependent variable, and f is a given non-zero function of the independent variables alone.
You already learned the basic philosophy for solving non-homogeneous linear equations in your study of elementary ordinary differential equations. Step one is to determine the general solution to the homogeneous equation. Step two is to find a particular solution to the non-homogeneous version. The general solution to the non-homogeneous equation is then obtained by adding the two together. Here is the general version of this procedure.

Theorem 2.3
Let u_p be a particular solution to the non-homogeneous linear equation

$L[u] = f$. Then any solution u to the non-homogenous equation can be written as the sum of $u_p + u_h$ where u_h is a solution to the homogenous equation.

Proof.
Let us first show that $u = u_p + u_h$ is also a solution to $L[u] = f$. By linearity,

$$L[u] = L[u_p + u_h] = L[u_p] + L[u_h] = f + 0 = f.$$

To show that every solution to the non-homogeneous equation can be expressed in this manner, suppose u satisfies $L[u] = f$. Set $w = u - u_p$. Then, by linearity,

$$L[w] = L[u - u_p] = L[u] - L[u_p] = 0,$$

and hence w is a solution to the homogeneous differential equation. Thus, $u = u_p + w$ ■

PDEs with Constraints
Also, as observed above, a linear partial differential equation has infinitely many solutions described by the family of solutions. In most applications, the family of solutions is of little use since it has to satisfy other supplementary conditions, usually called initial or boundary conditions. These conditions determine the unique solution of interest.
A **boundary value problem** is a partial differential equation where either the unknown function or its derivatives have values assigned on the physical boundary of the domain in which the problem is specified. These conditions are called **boundary conditions**. For example, the domain of the following problem is the square $[0,1] \times [0,1]$ with boundaries defined by $x = 0,\ x = 1$ for all $0 \le y \le 1$ and $y = 0,\ y = 1$ for all $0 \le x \le 1$.

$$\begin{aligned} u_{xx} + u_{yy} =& 0 && \text{if } 0 < x, y < 1 \\ u(x,0) = u(x,1) =& 0 && \text{if } 0 \le x \le 1 \\ u_x(0,y) = u_x(1,y) =& 0 && \text{if } 0 \le y \le 1. \end{aligned}$$

There are three types of boundary conditions which arise frequently in formulating physical problems:

1. **Dirichlet Boundary Conditions:** In this case, the dependent function u is prescribed on the boundary of the bounded domain. For example, if

the bounded domain is the rectangular plate $0 \leq x \leq L_1$ and $0 \leq y \leq L_2$, the boundary conditions $u(0, y), u(L_1, y), u(x, 0)$, and $u(x, L_2)$ are prescribed. The boundary conditions are called **homogeneous** if the dependent variable is zero at any point on the boundary, otherwise the boundary conditions are called **non-homogeneous.**

2. **Neumann Boundary Conditions:** In this case, first partial derivatives are prescribed on the boundary of the bounded domain. For example, the Neumann boundary conditions for a rod of length L, where $0 \leq x \leq L$, are of the form $u_x(0, t) = \alpha$ and $u_x(L, t) = \beta$, where α and β are constants.

3. **Robin Boundary Conditions:** This is a specification of a linear combination of the values of a function and the values of its derivative on the boundary of the domain. For example, if the physical domain if the interval $0 \leq x \leq L$ then an example of a Robin boundary condition could be $u_x(L, t) + \alpha u(L, t) = 0$.

An **initial value problem** is a partial differential equation together with a set of additional conditions on the unknown function or its derivatives at points in the given domain of the solution. These conditions are called **initial value conditions**. For example, the **transport equation**

$$\begin{aligned} u_t(x, t) + cu_x(x, t) =& 0 \\ u(x, 0) =& f(x). \end{aligned}$$

It can be shown that initial conditions for a linear PDE are sufficient for the existence of a unique solution.

We say that an initial and/or boundary value problem associated with a PDE is **well-posed** if it has a solution which is *unique* and depends continuously on the data given in the problem. The last condition, namely the continuous dependence is important in physical problems. This condition means that the solution changes by a small amount when the conditions change a little. Such solutions are said to be **stable**.

Example 2.6
For $x \in \mathbb{R}$ and $t > 0$ we consider the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} =& 0 \\ u(x,0) = u_t(x,0) =& 0. \end{aligned}$$

Clearly, $u(x,t) = 0$ is a solution to this problem.
(a) Let $0 < \epsilon << 1$ be a very small number. Show that the function $u_\epsilon(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$ is a solution to the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} =& 0 \\ u(x,0) =& 0 \\ u_t(x,0) =& \epsilon \sin\left(\frac{x}{\epsilon}\right). \end{aligned}$$

(b) Show that $\sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}, t > 0\} = \epsilon^2$. Thus, a small change in the initial data leads to a small change in the solution. Hence, the initial value problem is well-posed.

Solution.
(a) We have

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} =& \epsilon \sin\left(\frac{x}{\epsilon}\right) \cos\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial t^2} =& -\sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right) \\ \frac{\partial u_\epsilon}{\partial x} =& \epsilon \cos\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial x^2} =& -\sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right). \end{aligned}$$

Thus, $\frac{\partial^2 u_\epsilon}{\partial t^2} - \frac{\partial^2 u_\epsilon}{\partial x^2} = 0$. Moreover, $u_\epsilon(x,0) = 0$ and $\frac{\partial}{\partial t} u_\epsilon(x,0) = \epsilon \sin\left(\frac{x}{\epsilon}\right)$.
(b) We have

$$\begin{aligned} \sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}, t > 0\} =& \epsilon^2 \sup\left\{\left|\sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)\right| : x \in \mathbb{R}, t > 0\right\} \\ =& \epsilon^2 \blacksquare \end{aligned}$$

A problem that is not well-posed is referred to as an **ill-posed** problem. We illustrate this concept in the next example.

Example 2.7
For $x \in \mathbb{R}$ and $t > 0$ we consider the initial value problem

$$\begin{aligned} u_{tt} + u_{xx} =& 0 \\ u(x,0) = u_t(x,0) =& 0. \end{aligned}$$

Clearly, $u(x,t) = 0$ is a solution to this problem.
(a) Let $0 < \epsilon << 1$ be a very small number. Show that the function $u_\epsilon(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right)$, where

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

is a solution to the problem

$$\begin{aligned} u_{tt} + u_{xx} =& 0 \\ u(x,0) =& 0 \\ u_t(x,0) =& \epsilon \sin\left(\frac{x}{\epsilon}\right). \end{aligned}$$

(b) Show that $\sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}\} = \epsilon^2 \left|\sinh\left(\frac{t}{\epsilon}\right)\right|$.
(c) Find $\lim_{t\to\infty} \sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}\}$.

Solution.
(a) We have

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} =& \epsilon \sin\left(\frac{x}{\epsilon}\right) \cosh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial t^2} =& \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial u_\epsilon}{\partial x} =& \epsilon \cos\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial x^2} =& -\sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right). \end{aligned}$$

Thus, $\frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial^2 u_\epsilon}{\partial x^2} = 0$. Moreover, $u_\epsilon(x,0) = 0$ and $\frac{\partial}{\partial t} u_\epsilon(x,0) = \epsilon \sin\left(\frac{x}{\epsilon}\right)$.
(b) We have

$$\begin{aligned} \sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}\} =& \epsilon^2 \sup\{\left|\sinh\left(\frac{t}{\epsilon}\right) \sin\left(\frac{x}{\epsilon}\right)\right| : x \in \mathbb{R}\} \\ =& \epsilon^2 \left|\sinh\left(\frac{t}{\epsilon}\right)\right|. \end{aligned}$$

(c) We have

$$\lim_{t\to\infty}\sup\{|u_\epsilon(x,t)-u(x,t)|:x\in\mathbb{R}\}=\lim_{t\to\infty}\epsilon^2\left|\sinh\left(\frac{t}{\epsilon}\right)\right|=\infty.$$

Thus, a small change in the initial data leads to a catastrophically change in the solution. Hence, the given problem is ill-posed ■

Practice Problems

Problem 2.1
Determine a and b so that $u(x,y) = e^{ax+by}$ is a solution to the equation

$$u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0.$$

Problem 2.2
Consider the following differential equation

$$tu_{xx} - u_t = 0.$$

Suppose $u(t,x) = X(x)T(t)$. Show that there is a constant λ such that $X'' = \lambda X$ and $T' = \lambda tT$.

Problem 2.3
Consider the initial value problem

$$xu_x + (x+1)yu_y = 0, \quad x, y > 1$$

$$u(1,1) = e.$$

Show that $u(x,y) = \frac{xe^x}{y}$ is the solution to this problem.

Problem 2.4
Show that $u(x,y) = e^{-2y}\sin(x-y)$ is the solution to the initial value problem

$$\begin{cases} u_x + u_y + 2u = 0 & \text{for } x, y > 1 \\ u(x,0) = \sin x. \end{cases}$$

Problem 2.5
Solve each of the following differential equations:
(a) $\frac{du}{dx} = 0$ where $u = u(x)$.
(b) $\frac{\partial u}{\partial x} = 0$ where $u = u(x,y)$.

Problem 2.6
Solve each of the following differential equations:
(a) $\frac{d^2u}{dx^2} = 0$ where $u = u(x)$.
(b) $\frac{\partial^2 u}{\partial x \partial y} = 0$ where $u = u(x,y)$.

Problem 2.7
Show that $u(x,y) = f(y+2x) + xg(y+2x)$, where f and g are two arbitrary twice differentiable functions, satisfy the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$

Problem 2.8
Find the differential equation whose general solution is given by $u(x,t) = f(x-ct) + g(x+ct)$, where f and g are arbitrary twice differentiable functions in one variable.

Problem 2.9
Let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function in one variable. Prove that

$$u_t = p(u)u_x$$

has a solution satisfying $u(x,t) = f(x + p(u)t)$, where f is an arbitrary differentiable function. Then find the general solution to $u_t = (\sin u)u_x$.

Problem 2.10
Find the general solution to the pde

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Hint: See Problem 1.2.

Problem 2.11
Let $u(x,t)$ be a function such that u_{xx} exists and $u(0,t) = u(L,t) = 0$ for all $t \in \mathbb{R}$. Prove that

$$\int_0^L u_{xx}(x,t)u(x,t)dx \le 0.$$

Problem 2.12
Consider the initial value problem

$$u_t + u_{xx} = 0,\ x \in \mathbb{R},\ t > 0$$

$$u(x,0) = 1.$$

(a) Show that $u(x,t) \equiv 1$ is a solution to this problem.
(b) Show that $u_n(x,t) = 1 + \frac{e^{n^2 t}}{n} \sin nx$ is a solution to the initial value problem

$$u_t + u_{xx} = 0,\ x \in \mathbb{R},\ t > 0$$

$$u(x,0) = 1 + \frac{\sin nx}{n}.$$

(c) Find $\sup\{|u_n(x,0) - 1| : x \in \mathbb{R}\}$.
(d) Find $\sup\{|u_n(x,t) - 1| : x \in \mathbb{R}\}$.
(e) Show that the problem is ill-posed.

Problem 2.13
Find the general solution of each of the following PDEs by means of direct integration.
(a) $u_x = 3x^2 + y^2,\ u = u(x,y)$.
(b) $u_{xy} = x^2y,\ u = u(x,y)$.
(c) $u_{xyz} = 0,\ u = u(x,y,z)$.
(d) $u_{xtt} = e^{2x+3t},\ u = u(x,t)$.

Problem 2.14
Consider the second-order PDE

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

(a) Use the change of variables $v(x,y) = y - 2x$ and $w(x,y) = x$ to show that $u_{ww} = 0$.
(b) Find the general solution to the given PDE.

Problem 2.15
Derive the general solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

by using the change of variables $v = x + ct$ and $w = x - ct$.