# 19 Elliptic Type: Laplace's Equations in Rectangular Domains

Boundary value problems are of great importance in physical applications. Mathematically, a boundary-value problem consists of finding a function which satisfies a given partial differential equation and particular boundary conditions. Physically speaking, the problem is independent of time, involving only space coordinates.
Just as initial-value problems are associated with hyperbolic PDE, boundary value problems are associated with PDE of elliptic type. In contrast to initial-value problems, boundary-value problems are considerably more difficult to solve.
The main model example of an elliptic type PDE is the **Laplace equation**

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{19.1}$$

where the symbol $\Delta$ is referred to as the **Laplacian**. Solutions of this equation are called **harmonic functions.**

**Example 19.1**
Show that, for all $(x, y) \neq (0, 0)$, $u(x, y) = ax^2 - ay^2 + cx + dy + e$ is a harmonic function, where $a, b, c, d$, and $e$ are constants.

**Solution.**
We have

$$\begin{aligned} u_x =& 2ax + c \\ u_{xx} =& 2a \\ u_y =& -2ay + d \\ u_{yy} =& -2a. \end{aligned}$$

Plugging these expressions into the equation we find $u_{xx} + u_{yy} = 0$. Hence, $u(x, y)$ is harmonic ■

The Laplace equation is arguably the most important differential equation in all of applied mathematics. It arises in an astonishing variety of mathematical and physical systems, ranging through fluid mechanics, electromagnetism, potential theory, solid mechanics, heat conduction, geometry, probability,

number theory, and on and on.
There are two main modifications of the Laplace equation: the **Poisson equation** (a non-homogeneous Laplace equation):

$$\Delta u = f(x,y)$$

and the **eigenvalue problem** (the **Helmholtz equation**):

$$\Delta u = \lambda u, \quad \lambda \in \mathbb{R}.$$

**Solving Laplace's Equation (19.1)**
Note first that both independent variables are spatial variables and each variable occurs in a 2nd order derivative and so we will need two boundary conditions for each variable a total of four boundary conditions.
Consider (19.1) in the rectangle

$$\Omega = \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq b\}$$

with the Dirichlet boundary conditions

$$u(0,y) = f_1(y),\ u(a,y) = f_2(y),\ u(x,0) = g_1(x),\ u(x,b) = g_2(x)$$

where $0 \leq x \leq a$ and $0 \leq y \leq b$.
The separation of variables method is most successful when the boundary conditions are homogeneous. Thus, solving the Laplace's equation in $\Omega$ requires solving four initial boundary conditions problems, where in each problem three of the four conditions are homogeneous. The four problems to be solved are

$$(I)\left\{\begin{array}{c} u_{xx}+u_{yy}=0 \\ u(0,y)=f_1(y), \\ u(a,y)=u(x,0)=u(x,b)=0 \end{array}\right. \qquad (II)\left\{\begin{array}{c} u_{xx}+u_{yy}=0 \\ u(a,y)=f_2(y), \\ u(0,y)=u(x,0)=u(x,b)=0 \end{array}\right.$$

$$(III)\left\{\begin{array}{c} u_{xx}+u_{yy}=0 \\ u(x,0)=g_1(x), \\ u(0,y)=u(a,y)=u(x,b)=0 \end{array}\right. \qquad (IV)\left\{\begin{array}{c} u_{xx}+u_{yy}=0 \\ u(x,b)=g_2(x), \\ u(0,y)=u(a,y)=u(x,0)=0. \end{array}\right.$$

If we let $u_i(x,y)$, $i = 1,2,3,4$, denote the solution of each of the above problems, then the solution to our original system will be

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

In each of the above problems, we will apply separation of variables to (19.1) and find a product solution that will satisfy the differential equation and the three homogeneous boundary conditions. Using the Principle of Superposition we will find a solution to the problem and then apply the final boundary condition to determine the value of the constant(s) that are left in the problem. The process is nearly identical in many ways to what we did when we were solving the heat equation.
We will illustrate how to find $u(x,y)=u_4(x,y)$. So let's assume that the solution can be written in the form $u(x,y)=X(x)Y(y)$. Substituting in (19.1), we obtain

$$X''(x)Y(y)+X(x)Y''(y)=0.$$

Assuming $X(x)Y(y)$ is nonzero, that is $u$ is the non-trivial solution. Dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of $x$ while the right hand side is a function of $y$. This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\lambda$$

where $\lambda$ is a constant. This results in the following two ODEs

$$X''-\lambda X=0 \text{ and } Y''+\lambda Y=0.$$

As far as the boundary conditions, we have for all $0\le x\le a$ and $0\le y\le b$

$$u(0,y)=0=X(0)Y(y)\Longrightarrow X(0)=0$$

$$u(a,y)=0=X(a)Y(y)\Longrightarrow X(a)=0$$
$$u(x,0)=0=X(x)Y(0)\Longrightarrow Y(0)=0$$
$$u(x,b)=g_2(x)=X(x)Y(b).$$

Note that $X$ and $Y$ are not the zero functions for otherwise $u\equiv 0$ and this contradicts our assumption that $u$ is the non-trivial solution.
Consider the first equation: since $X''-\lambda X=0$ the solution depends on the sign of $\lambda$. If $\lambda=0$ then $X(x)=Ax+B$. Now, the conditions $X(0)=X(a)=0$

imply $A = B = 0$ and so $u \equiv 0$. So assume that $\lambda \neq 0$. If $\lambda > 0$ then $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Now, the conditions $X(0) = X(a) = 0,\ \lambda \neq 0$ imply $A = B = 0$ and hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have $\lambda < 0$. In this case,

$$X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x.$$

The condition $X(0) = 0$ implies $A = 0$. The condition $X(a) = 0$ implies $B\sin\sqrt{-\lambda}a = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin\sqrt{-\lambda}a = 0$ or $\sqrt{-\lambda}a = n\pi$ where $n \in \mathbb{N}$. Solving for $\lambda$ we find $\lambda_n = -\frac{n^2\pi^2}{a^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = \sin\frac{n\pi}{a}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$Y'' + \lambda Y = 0$$

we obtain

$$Y_n(y) = a_n e^{\sqrt{-\lambda_n}y} + b_n e^{-\sqrt{-\lambda_n}y} = A_n\cosh\sqrt{-\lambda_n}y + B_n\sinh\sqrt{-\lambda_n}y,\ n \in \mathbb{N}.$$

Using the boundary condition $Y(0) = 0$ we obtain $A_n = 0$ for all $n \in \mathbb{N}$. Hence, the functions

$$u_n(x,y) = B_n\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right),\ n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions $u(0,y) = u(a,y) = u(x,0) = 0$. Now, in order for these solutions to satisfy the boundary value condition $u(x,b) = g_2(x)$, we invoke the superposition principle of linear PDE to write

$$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right). \tag{19.2}$$

To determine the unknown constants $B_n$ we use the boundary condition $u(x,b) = g_2(x)$ in (19.2) to obtain

$$g_2(x) = \sum_{n=1}^{\infty}\left(B_n\sinh\left(\frac{n\pi}{a}b\right)\right)\sin\left(\frac{n\pi}{a}x\right).$$

Since the right-hand side is the Fourier sine series of $g_2(x)$ on the interval $[0, a]$, the coefficients $B_n$ are given by

$$B_n = \left[\frac{2}{a}\int_0^a g_2(x)\sin\left(\frac{n\pi}{a}x\right)dx\right]\left[\sinh\left(\frac{n\pi}{a}b\right)\right]^{-1}. \tag{19.3}$$

Thus, the solution to the Laplace's equation is given by (19.2) with the $B_n'$s calculated from (19.3).

**Example 19.2**
Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = f_1(y), \\ u(a, y) = u(x, 0) = u(x, b) = 0. \end{cases}$$

**Solution.**
Assume that the solution can be written in the form $u(x, y) = X(x)Y(y)$. Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of $x$ while the right hand side is a function of $y$. This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where $\lambda$ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all $0 \le x \le a$ and $0 \le y \le b$

$$u(0, y) = f_1(y) = X(0)Y(y)$$

$$u(a, y) = 0 = X(a)Y(y) \Longrightarrow X(a) = 0$$

$$u(x,0) = 0 = X(x)Y(0) \Longrightarrow Y(0) = 0$$
$$u(x,b) = 0 = X(x)Y(b) \Longrightarrow Y(b) = 0.$$

Note that $X$ and $Y$ are not the zero functions for otherwise $u \equiv 0$ and this contradicts our assumption that $u$ is the non-trivial solution.
Consider the second equation: since $Y'' + \lambda Y = 0$ the solution depends on the sign of $\lambda$. If $\lambda = 0$ then $Y(y) = Ay + B$. Now, the conditions $Y(0) = Y(b) = 0$ imply $A = B = 0$ and so $u \equiv 0$. So assume that $\lambda \neq 0$. If $\lambda < 0$ then $Y(y) = Ae^{\sqrt{-\lambda}y} + Be^{-\sqrt{-\lambda}y}$. Now, the condition $Y(0) = Y(b) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have $\lambda > 0$. In this case,

$$Y(y) = A\cos\sqrt{\lambda}y + B\sin\sqrt{\lambda}y.$$

The condition $Y(0) = 0$ implies $A = 0$. The condition $Y(b) = 0$ implies $B\sin\sqrt{\lambda}b = 0$. We must have $B \neq 0$ otherwise $Y(y) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin\sqrt{\lambda}b = 0$ or $\sqrt{\lambda}b = n\pi$ where $n \in \mathbb{N}$. Solving for $\lambda$ we find $\lambda_n = \frac{n^2\pi^2}{b^2}$. Thus, we obtain infinitely many solutions given by

$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right), \quad n \in \mathbb{N}.$$

Now, solving the equation

$$X'' - \lambda X = 0, \ \lambda > 0$$

we obtain

$$X_n(x) = a_n e^{\sqrt{\lambda_n}x} + b_n e^{-\sqrt{\lambda_n}x} = A_n\cosh\left(\frac{n\pi}{b}x\right) + B_n\sinh\left(\frac{n\pi}{b}x\right), \ n \in \mathbb{N}.$$

However, this is not really suited for dealing with the boundary condition $X(a) = 0$. So, let's also notice that the following is also a solution.

$$X_n(x) = A_n\cosh\left(\frac{n\pi}{b}(x-a)\right) + B_n\sinh\left(\frac{n\pi}{b}(x-a)\right), \ n \in \mathbb{N}.$$

Now, using the boundary condition $X(a) = 0$ we obtain $A_n = 0$ for all $n \in \mathbb{N}$. Hence, the functions

$$u_n(x,y) = B_n\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{b}(x-a)\right), \ n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions $u(a,y) = u(x,0) = u(x,b) = 0$. Now, in order for these solutions to satisfy the boundary value condition $u(0,y) = f_1(y)$, we invoke the superposition principle of linear PDE to write

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{b}(x-a)\right). \tag{19.4}$$

To determine the unknown constants $B_n$ we use the boundary condition $u(0,y) = f_1(y)$ in (19.4) to obtain

$$f_1(y) = \sum_{n=1}^{\infty}\left(B_n \sinh\left(-\frac{n\pi}{b}a\right)\right)\sin\left(\frac{n\pi}{b}y\right).$$

Since the right-hand side is the Fourier sine series of $f_1(y)$ on the interval $[0,b]$, the coefficients $B_n$ are given by

$$B_n = \left[\frac{2}{b}\int_0^b f_1(y)\sin\left(\frac{n\pi}{b}y\right)dy\right]\left[\sinh\left(-\frac{n\pi}{b}a\right)\right]^{-1}. \tag{19.5}$$

Thus, the solution to the Laplace's equation is given by (19.4) with the $B_n'$s calculated from (19.5) ■

**Example 19.3**
Solve

$$\begin{gathered} u_{xx} + u_{yy} = 0, \quad 0 < x < L,\ 0 < y < H \\ u(0,y) = u(L,y) = 0, \quad 0 < y < H \\ u(x,0) = u_y(x,0), \quad u(x,H) = f(x), \quad 0 < x < L. \end{gathered}$$

**Solution.**
Using separation of variables we find

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

We first solve

$$\left\{\begin{array}{ll} X'' - \lambda X = 0 & 0 < x < L \\ X(0) = X(L) = 0. & \end{array}\right.$$

We find $\lambda_n = -\frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = \sin\frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Next we need to solve

$$\begin{cases} Y'' + \lambda Y = 0 & 0 < y < H \\ Y(0) - Y'(0) = 0. \end{cases}$$

The solution of the ODE is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{L} y\right) + B_n \sinh\left(\frac{n\pi}{L} y\right),\ n \in \mathbb{N}.$$

The boundary condition $Y(0) - Y'(0) = 0$ implies

$$A_n - B_n \frac{n\pi}{L} = 0.$$

Hence,

$$Y_n = B_n \frac{n\pi}{L} \cosh\left(\frac{n\pi}{L} y\right) + B_n \sinh\left(\frac{n\pi}{L} y\right),\ n \in \mathbb{N}.$$

Using the superposition principle and the results above we have

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L} y\right) + \sinh\left(\frac{n\pi}{L} y\right)\right].$$

Substituting in the condition $u(x,H) = f(x)$ we find

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L} H\right) + \sinh\left(\frac{n\pi}{L} H\right)\right].$$

Recall the Fourier sine series of $f$ on $[0, L]$ given by

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx.$$

Thus, the general solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L} y\right) + \sinh\left(\frac{n\pi}{L} y\right)\right].$$

with the $B_n$ satisfying

$$B_n \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L} H\right) + \sinh\left(\frac{n\pi}{L} H\right)\right] = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx \ \blacksquare$$

# Practice Problems

**Problem 19.1**

Solve

$$\left\{\begin{array}{c} u_{xx}+u_{yy}=0 \\ u(a,y)=f_2(y), \\ u(0,y)=u(x,0)=u(x,b)=0. \end{array}\right.$$

**Problem 19.2**

Solve

$$\left\{\begin{array}{c} u_{xx}+u_{yy}=0 \\ u(x,0)=g_1(x), \\ u(0,y)=u(a,y)=u(x,b)=0. \end{array}\right.$$

**Problem 19.3**

Solve

$$\left\{\begin{array}{c} u_{xx}+u_{yy}=0 \\ u(x,0)=u(0,y)=0, \\ u(1,y)=2y, u(x,1)=3\sin \pi x+2x \end{array}\right.$$

where $0 \le x \le 1$ and $0 \le y \le 1$. Hint: Define $U(x,y)=u(x,y)-2xy$.

**Problem 19.4**

Show that $u(x,y)=x^2-y^2$ and $u(x,y)=2xy$ are harmonic functions.

**Problem 19.5**

Solve

$$u_{xx}+u_{yy}=0, \quad 0 \le x \le L, \quad -\frac{H}{2} \le y \le \frac{H}{2}$$

subject to

$$u(0,y)=u(L,y)=0, \quad -\frac{H}{2} < y < \frac{H}{2}$$

$$u(x,-\frac{H}{2})=f_1(x), \quad u(x,\frac{H}{2})=f_2(x), \ 0 \le x \le L.$$

**Problem 19.6**

Consider a complex valued function $f(z)=u(x,y)+iv(x,y)$ where $i=\sqrt{-1}$. We say that $f$ is **holomorphic** or **analytic** if and only if $f$ can be expressed as a power series in $z$, i.e.

$$u(x,y)+iv(x,y)=\sum_{n=0}^{\infty} a_n z^n.$$

(a) By differentiating with respect to $x$ and $y$ show that

$$u_x = v_y \text{ and } u_y = -v_x$$

These are known as the Cauchy-Riemann equations.
(b) Show that $\Delta u = 0$ and $\Delta v = 0$.

**Problem 19.7**
Show that Laplace's equation in polar coordinates is given by

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

**Problem 19.8**
Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le 2, \ 0 \le y \le 3$$

subject to

$$u(x,0) = 0, \quad u(x,3) = \frac{x}{2}$$

$$u(0,y) = \sin\left(\frac{4\pi}{3}y\right), \quad u(2,y) = 7.$$

**Problem 19.9**
Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le L, \ 0 \le y \le H$$

subject to

$$u_y(x,0) = 0, \quad u(x,H) = 0$$

$$u(0,y) = u(L,y) = 4\cos\left(\frac{\pi y}{2H}\right).$$

**Problem 19.10**
Solve

$$u_{xx} + u_{yy} = 0, \quad x > 0, \ 0 \le y \le H$$

subject to

$$u(0,y) = f(y), |u(x,0)| < \infty$$

$$u_y(x,0) = u_y(x,H) = 0.$$

**Problem 19.11**
Consider Laplace's equation inside a rectangle

$$u_{xx} + u_{yy} = 0, \ 0 \le x \le L, \ 0 \le y \le H$$

subject to the boundary conditions

$$u(0,y) = 0, \ u(L,y) = 0, \ u(x,0) - u_y(x,0) = 0, \ u(x,H) = 20\sin\left(\frac{\pi x}{L}\right) - 5\sin\left(\frac{3\pi x}{L}\right).$$

Find the solution $u(x,y)$.

**Problem 19.12**
Solve Laplace'e equation $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x, y < 1$ subject to the conditions

$$u(0,y) = u(1,y) = 0, \quad 0 < y < 1$$
$$u(x,0) = \sin(2\pi x), \quad u_y(x,0) = -2\pi\sin(2\pi x), \quad 0 < x < 1.$$

**Problem 19.13**
Find the solution to Laplace's equation on the rectangle $0 < x < 1, 0 < y < 1$ with boundary conditions

$$u(x,0) = 0, \quad u(x,1) = 1$$

$$u_x(0,y) = u_x(1,y) = 0.$$

**Problem 19.14**
Solve Laplace's equation on the rectangle $0 < x < a, \ 0 < y < b$ with the boundary conditions

$$u_x(0,y) = -a, \quad u_x(a,y) = 0$$

$$u_y(x,0) = b, \quad u_y(x,b) = 0.$$

**Problem 19.15**
Solve Laplace's equation on the rectangle $0 < x < \pi, \ 0 < y < 2$ with the boundary conditions

$$u(0,y) = u(\pi,y) = 0$$
$$u_y(x,0) = 0, \quad u_y(x,2) = 2\sin 3x - 5\sin 10x.$$