18 Solutions of the Heat Equation by the Separation of Variables Method

In this section we apply the method of separation of variables in solving the one spatial dimension of the heat equation.

The Heat Equation with Dirichlet Boundary Conditions
Consider the problem of finding all nontrivial solutions to the heat equation $u_t = ku_{xx}$ that satisfies the initial time condition $u(x,0) = f(x)$ and the Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$ (that is, the endpoints are assumed to be at zero temperature) with u not the trivial solution. Let's assume that the solution can be written in the form $u(x,t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the LHS only depends on x and the RHS only depends on t, there must be a constant λ such that

$$\tfrac{X''}{X} = \lambda \text{ and } \tfrac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0,t) = 0 = X(0)T(t) \Longrightarrow X(0) = 0$$

and

$$u(L,t) = 0 = X(L)T(t) \Longrightarrow X(L) = 0.$$

Note that T is not the zero function for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.
Next, we consider the three cases of the sign of λ.

Case 1: $\lambda = 0$
In this case, $X'' = 0$. Solving this equation we find $X(x) = ax + b$. Since $X(0) = 0$ we find $b = 0$. Since $X(L) = 0$ we find $a = 0$. Hence, $X \equiv 0$ and $u(x,t) \equiv 0$. That is, u is the trivial solution.

Case 2: $\lambda > 0$
In this case, $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Again, the conditions $X(0) = X(L) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution.

Case 3: $\lambda < 0$
In this case, $X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$. The condition $X(0) = 0$ implies $A = 0$. The condition $X(L) = 0$ implies $B\sin\sqrt{-\lambda}L = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin\sqrt{-\lambda}L = 0$ or $\sqrt{-\lambda}L = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda = -\frac{n^2\pi^2}{L^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = A_n \sin\frac{n\pi}{L}x, \quad n = 1, 2, \cdots .$$

Now, solving the equation

$$T' - \lambda k T = 0$$

by the method of separation of variables we obtain

$$T_n(t) = B_n e^{-\frac{n^2\pi^2}{L^2}kt}, \; n = 1, 2, \cdots$$

Hence, the functions

$$u_n(x,t) = C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}, \; n = 1, 2, \cdots$$

satisfy $u_t = ku_{xx}$ and the boundary conditions $u(0,t) = u(L,t) = 0$.
Now, in order for these solutions to satisfy the initial value condition $u(x,0) = f(x)$, we invoke the superposition principle of linear PDE to write

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}. \tag{18.1}$$

To determine the unknown constants C_n we use the initial condition $u(x,0) = f(x)$ in (18.1) to obtain

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right).$$

Since the right-hand side is the Fourier sine series of f on the interval $[0, L]$, the coefficients C_n are given by

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx. \tag{18.2}$$

Thus, the solution to the heat equation is given by (18.1) with the C_n's calculated from (18.2).

The Heat Equation with Neumann Boundary Conditions

When both ends of the bar are insulated, that is, there is no heat flow out of them, we use the boundary conditions

$$u_x(0,t) = u_x(L,t) = 0.$$

In this case, the general form of the heat equation initial boundary value problem is to find $u(x,t)$ satisfying

$$\begin{aligned} u_t(x,t) =& ku_{xx}(x,t), \quad 0 \le x \le L,\ t > 0 \\ u(x,0) =& f(x), \quad 0 \le x \le L \\ u_x(0,t) =& u_x(L,t) = 0, \quad t > 0. \end{aligned}$$

Since $0 = u_x(0,t) = X'(0)T(t)$ we obtain $X'(0) = 0$. Likewise, $0 = u_x(L,t) = X'(L)T(t)$ implies $X'(L) = 0$. We again consider the following three cases:
• If $\lambda = 0$ then $X(x) = A + Bx$. Since $X'(0) = 0$, we find $B = 0$. Thus, $X(x) = A$ and $T(t) =$ constant so that $u(x,t) =$ constant which is impossible if $f(x)$ is not the constant function.
• If $\lambda > 0$ then a simple calculation shows that $u(x,t)$ is the trivial solution. Again, because of the condition $u(x,0) = f(x)$, this solution is discarded.
• If $\lambda < 0$ then $X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$ and upon differentiation with respect to x we find

$$X'(x) = -\sqrt{-\lambda}A\sin\sqrt{-\lambda}x + \sqrt{-\lambda}B\cos\sqrt{-\lambda}x.$$

The conditions $X'(0) = X'(L) = 0$ imply $\sqrt{-\lambda}B = 0$ and $\sqrt{-\lambda}A\sin\sqrt{-\lambda}L = 0$. Hence, $B = 0$ and $\lambda = -\frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = A_n\cos\left(\frac{n\pi}{L}x\right),\ n = 1, 2, \cdots$$

and

$$u_n(x,t) = C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

By the superposition principle, the required solution to the heat equation with Neumann boundary conditions is given by

$$\overline{u}(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

In order to satisfy the initial condition $u(x,0) = f(x)$, we let

$$u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

where

$$C_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \;\; n = 0, 1, 2, \cdots .$$

Practice Problems

Problem 18.1
Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $f(x) = \sin\left(\frac{\pi}{2}x\right) + 3\sin\left(\frac{5\pi}{2}x\right)$.

Problem 18.2
Find the temperature in a homogeneous bar of heat conducting material of length L with its end points kept at zero and initial temperature distribution given by $f(x) = \frac{xd}{L^2}(L-x),\ 0 \le x \le L$.

Problem 18.3
Find the temperature in a thin metal rod of length L, with both ends insulated (so that there is no passage of heat through the ends) and with initial temperature in the rod $f(x) = \sin\left(\frac{\pi}{L}x\right)$.

Problem 18.4
Solve the following heat equation with Dirichlet boundary conditions

$$u_t = ku_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = \begin{cases} 1 & 0 \le x < \frac{L}{2} \\ 2 & \frac{L}{2} \le x \le L. \end{cases}$$

Problem 18.5
Solve

$$u_t = ku_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = 6\sin\left(\frac{9\pi}{L}x\right).$$

Problem 18.6
Solve

$$u_t = ku_{xx}$$

subject to

$$u_x(0,t) = u_x(L,t) = 0$$

$$u(x,0) = \begin{cases} 0 & 0 \le x < \frac{L}{2} \\ 1 & \frac{L}{2} \le x \le L. \end{cases}$$

Problem 18.7
Solve

$$u_t = ku_{xx}$$

subject to

$$u_x(0,t) = u_x(L,t) = 0$$
$$u(x,0) = 6 + 4\cos\left(\frac{3\pi}{L}x\right).$$

Problem 18.8
Solve

$$u_t = ku_{xx}$$

subject to

$$u_x(0,t) = u_x(L,t) = 0$$
$$u(x,0) = -3\cos\left(\frac{8\pi}{L}x\right).$$

Problem 18.9
Find the general solution $u(x,t)$ of

$$u_t = u_{xx} - u, \quad 0 < x < L, \ t > 0$$
$$u_x(0,t) = 0 = u_x(L,t), \quad t > 0.$$

Briefly describe its behavior as $t \to \infty$.

Problem 18.10 *(Energy method)*
Let u_1 and u_2 be two solutions to the Neumann boundary value problem

$$u_t = u_{xx} - u, \quad 0 < x < 1, \ t > 0$$
$$u_x(0,t) = u_x(1,t) = 0, \quad t > 0$$
$$u(x,0) = g(x), \quad 0 < x < 1$$

Define $w(x,t) = u_1(x,t) - u_2(x,t)$.
(a) Show that w satisfies the initial value problem

$$w_t = w_{xx} - w, \quad 0 < x < 1, \ t > 0$$
$$w_x(0,t) = w_x(1,t) = w(x,0) = 0, \quad 0 < x < 1, t > 0$$

(b) Define $E(t) = \int_0^1 w^2(x,t)dx \geq 0$ for all $t \geq 0$. Show that $E'(t) \leq 0$. Hence, $0 \leq E(t) \leq E(0)$ for all $t > 0$.
(c) Show that $E(t) = 0, \ w(x,t) = 0$. Hence, conclude that $u_1 = u_2$.

Problem 18.11
Consider the heat induction in a bar where the left end temperature is maintained at 0, and the right end is perfectly insulated. We assume $k = 1$ and $L = 1$.
(a) Derive the boundary conditions of the temperature at the endpoints.
(b) Following the separation of variables approach, derive the ODEs for X and T.
(c) Consider the equation in $X(x)$. What are the values of $X(0)$ and $X'(1)$? Show that solutions of the form $X(x) = \sin\sqrt{-\lambda}x$ satisfy the ODE and one of the boundary conditions. Can you choose a value of λ so that the other boundary condition is also satisfied?

Problem 18.12
Using the method of separation of variables find the solution of the heat equation

$$u_t = ku_{xx}$$

satisfying the following boundary and initial conditions:
(a) $u(0,t) = u(L,t) = 0,\ u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$
(b) $u(0,t) = u(L,t) = 0,\ u(x,0) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$

Problem 18.13
Using the method of separation of variables find the solution of the heat equation

$$u_t = ku_{xx}$$

satisfying the following boundary and initial conditions:
(a) $u_x(0,t) = u_x(L,t) = 0,\ u(x,0) = \cos\left(\frac{\pi x}{L}\right) + 4\cos\left(\frac{5\pi x}{L}\right)$.
(b) $u_x(0,t) = u_x(L,t) = 0,\ u(x,0) = 5$.

Problem 18.14
Find the solution of the following heat conduction partial differential equation

$$u_t = 8u_{xx},\quad 0 < x < 4\pi,\quad t > 0$$

$$u(0,t) = u(4\pi,t) = 0,\quad t > 0$$

$$u(x,0) = 6\sin x,\quad 0 < x < 4\pi.$$