17 Separation of Variables for PDEs

Finding analytic solutions to PDEs is essentially impossible. Most of the PDE techniques involve a mixture of analytic, qualitative and numeric approaches. Of course, there are some easy PDEs too. If you are lucky your PDE has a solution with separable variables. In this chapter we discuss the application of the method of separation of variables in the solution of PDEs.

17.1 Second Order Linear Homogenous ODE with Constant Coefficients

In this section, we review the basics of finding the general solution to the ODE

$$ay'' + by' + cy = 0 \tag{17.1}$$

where a, b, and c are constants. The process starts by solving the **characteristic equation**

$$ar^2 + br + c = 0$$

which is a quadratic equation with roots

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We consider the following three cases:

• If $b^2 - 4ac > 0$ then the general solution to (17.1) is given by

$$y(t) = Ae^{\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)t} + Be^{\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)t}.$$

• If $b^2 - 4ac = 0$ then the general solution to (17.1) is given by

$$y(t) = Ae^{-\frac{b}{2a}t} + Bte^{-\frac{b}{2a}t}.$$

• If $b^2 - 4ac < 0$ then

$$r_{1,2} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

and the general solution to (17.1) is given by

$$y(t) = Ae^{-\frac{b}{2a}t} \cos\left(\frac{\sqrt{4ac - b^2}}{2a}\right)t + Be^{-\frac{b}{2a}t} \sin\left(\frac{\sqrt{4ac - b^2}}{2a}\right)t.$$

17.2 The Method of Separation of Variables for PDEs

In developing a solution to a partial differential equation by separation of variables, one assumes that it is possible to separate the contributions of the independent variables into separate functions that each involve only one independent variable. Thus, the method consists of the following steps
1. Factorize the (unknown) dependent variable of the PDE into a product of functions, each of the factors being a function of one independent variable. That is,

$$u(x, y) = X(x)Y(y).$$

2. Substitute into the PDE, and divide the resulting equation by $X(x)Y(y)$.
3. Then the problem turns into a set of separated ODEs (one for $X(x)$ and one for $Y(y)$.)
4. The general solution of the ODEs is found, and boundary initial conditions are imposed.
5. $u(x, y)$ is formed by multiplying together $X(x)$ and $Y(y)$.

We illustrate these steps in the next three examples.

Example 17.1
Find all the solutions of the form $u(x, t) = X(x)T(t)$ of the equation

$$u_{xx} - u_x = u_t.$$

Solution.
It is very easy to find the derivatives of a separable function:

$$u_x = X'(x)T(t), u_t = X(x)T'(t) \text{ and } u_{xx} = X''(x)T(t),$$

which is basically a consequence of the fact that differentiation with respect to x sees t as a constant, and vice versa. Now, the equation $u_{xx} - u_x = u_t$ becomes

$$X''(x)T(t) - X'(x)T(t) = X(x)T'(t).$$

We can separate variables further. Division by $X(x)T(t)$ gives

$$\frac{X''(x) - X'(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

The expression on the LHS is a function of x whereas the one on the RHS is a function of t only. They both have to be constant. That is,

$$\frac{X''(x) - X'(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda.$$

Thus, we have the following ODEs:

$$X'' - X' - \lambda X = 0 \text{ and } T' = \lambda T.$$

The second equation is easy to solve: $T(t) = Ce^{\lambda t}$. The first equation is solved via the characteristic equation $\omega^2 - \omega - \lambda = 0$, whose solutions are

$$\omega = \frac{1 \pm \sqrt{1+4\lambda}}{2}.$$

If $\lambda > -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{1+\sqrt{1+4\lambda}}{2}x} + Be^{\frac{1-\sqrt{1+4\lambda}}{2}x}.$$

In this case,

$$u(x,t) = De^{\frac{1+\sqrt{1+4\lambda}}{2}x}e^{\lambda t} + Ee^{\frac{1-\sqrt{1+4\lambda}}{2}x}e^{\lambda t}.$$

If $\lambda = -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{x}{2}} + Bxe^{\frac{x}{2}}$$

and in this case

$$u(x,t) = (D + Ex)e^{\frac{x}{2} - \frac{t}{4}}.$$

If $\lambda < -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{x}{2}}\cos\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) + Be^{\frac{x}{2}}\sin\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right).$$

In this case,

$$u(x,t) = D'e^{\frac{x}{2}+\lambda t}\cos\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) + B'e^{\frac{x}{2}+\lambda t}\sin\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) \blacksquare$$

Example 17.2
Solve Laplace's equation using the separation of variables method

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Solution.
We look for a solution of the form $u(x,y) = X(x)Y(y)$. Substituting in the Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of x while the right hand side is a function of y. This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

The solutions of these equations depend on the sign of λ.
• If $\lambda > 0$ then the solutions are given

$$\begin{aligned} X(x) =& Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ Y(y) =& C\cos\sqrt{\lambda}y + D\sin\sqrt{\lambda}y \end{aligned}$$

where A, B, C, and D are constants. In this case,

$$\begin{aligned} u(x,t) =& k_1e^{\sqrt{\lambda}x}\cos\sqrt{\lambda}y + k_2e^{\sqrt{\lambda}x}\sin\sqrt{\lambda}y \\ &+ k_3e^{-\sqrt{\lambda}x}\cos\sqrt{\lambda}y + k_4e^{-\sqrt{\lambda}x}\sin\sqrt{\lambda}y. \end{aligned}$$

• If $\lambda = 0$ then

$$\begin{aligned} X(x) =& Ax + B \\ Y(y) =& Cy + D \end{aligned}$$

where A, B, and C are arbitrary constants. In this case,

$$u(x,y) = k_1xy + k_2x + k_3y + k_4.$$

• If $\lambda < 0$ then

$$\begin{aligned} X(x) =& A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x \\ Y(y) =& Ce^{\sqrt{-\lambda}y} + De^{-\sqrt{-\lambda}y} \end{aligned}$$

where A, B, C, and D are arbitrary constants. In this case,

$$\begin{aligned} u(x,y) =& k_1\cos\sqrt{-\lambda}xe^{\sqrt{-\lambda}y} + k_2\cos\sqrt{-\lambda}xe^{-\sqrt{-\lambda}y} \\ &+ k_3\sin\sqrt{-\lambda}xe^{\sqrt{-\lambda}y} + k_4\sin\sqrt{-\lambda}xe^{-\sqrt{-\lambda}y} \blacksquare \end{aligned}$$

Example 17.3
Solve using the separation of variables method.

$$yu_x - xu_y = 0.$$

Solution.
Substitute $u(x,t) = X(x)Y(y)$ into the given equation we find

$$yX'Y - xXY' = 0.$$

This can be separated into

$$\frac{X'}{xX} = \frac{Y'}{yY}.$$

The left hand side is a function of x while the right hand side is a function of y. This says that they must equal to a constant. That is,

$$\frac{X'}{xX} = \frac{Y'}{yY} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X' - \lambda xX = 0 \text{ and } Y' - \lambda yY = 0.$$

Solving these equations using the method of separation of variable for ODEs we find $X(x) = Ae^{\frac{\lambda x^2}{2}}$ and $Y(y) = Be^{\frac{\lambda y^2}{2}}$. Thus, the general solution is given by

$$u(x,y) = Ce^{\frac{\lambda(x^2+y^2)}{2}} \blacksquare$$

Practice Problems

Problem 17.1
Solve using the separation of variables method

$$\Delta u + \lambda u = 0.$$

Problem 17.2
Solve using the separation of variables method

$$u_t = ku_{xx}.$$

Problem 17.3
Derive the system of ordinary differential equations for $R(r)$ and $\Theta(\theta)$ that is satisfied by solutions to

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Problem 17.4
Derive the system of ordinary differential equations and boundary conditions for $X(x)$ and $T(t)$ that is satisfied by solutions to

$$u_{tt} = u_{xx} - 2u, \quad 0 < x < 1, \ t > 0$$

$$u(0,t) = 0 = u_x(1,t) \quad t > 0$$

of the form $u(x,t) = X(x)T(t)$. (Note: you do not need to solve for X and T.)

Problem 17.5
Derive the system of ordinary differential equations and boundary conditions for $X(x)$ and $T(t)$ that is satisfied by solutions to

$$u_t = ku_{xx}, \quad 0 < x < L, \ t > 0$$

$$u(x,0) = f(x), \ u(0,t) = 0 = u_x(L,t) \quad t > 0$$

of the form $u(x,t) = X(x)T(t)$. (Note: you do not need to solve for X and T.)

Problem 17.6
Find all product solutions of the PDE $u_x + u_t = 0$.

Problem 17.7
Derive the system of ordinary differential equations for $X(x)$ and $Y(y)$ that is satisfied by solutions to

$$3u_{yy} - 5u_{xxxy} + 7u_{xxy} = 0.$$

of the form $u(x,y) = X(x)Y(y)$.

Problem 17.8
Find the general solution by the method of separation of variables.

$$u_{xy} + u = 0.$$

Problem 17.9
Find the general solution by the method of separation of variables.

$$u_x - yu_y = 0.$$

Problem 17.10
Find the general solution by the method of separation of variables.

$$u_{tt} - u_{xx} = 0.$$

Problem 17.11
For the following PDEs find the ODEs implied by the method of separation of variables.
(a) $u_t = kr(ru_r)_r$
(b) $u_t = ku_{xx} - \alpha u$
(c) $u_t = ku_{xx} - au_x$
(d) $u_{xx} + u_{yy} = 0$
(e) $u_t = ku_{xxxx}$.

Problem 17.12
Find all solutions to the following partial differential equation that can be obtained via the separation of variables.

$$u_x - u_y = 0.$$

Problem 17.13
Separate the PDE $u_{xx} - u_y + u_{yy} = u$ into two ODEs with a parameter. You do not need to solve the ODEs.