16 Fourier Sines Series and Fourier Cosines Series

In this section we discuss some important properties of Fourier series when the underlying function f is either even or odd.
A function f is **odd** if it satisfies $f(-x) = -f(x)$ for all x in the domain of f whereas f is **even** if it satisfies $f(-x) = f(x)$ for all x in the domain of f.

Example 16.1
Show the following
(a) If f and g are either both even or both odd then fg is even.
(b) If f is odd and g is even then fg is odd.

Solution.
(a) Suppose that both f and g are even. Then $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$. That is, fg is even. Now, suppose that both f and g are odd. Then $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = (fg)(x)$. That is, fg is even.
(b) f is odd and g is even. Then $(fg)(-x) = f(-x)g(-x) = -f(x)g(x) = -(fg)(x)$. That is, fg is odd ■

Example 16.2
(a) Show that for any even function $f(x)$ defined on the interval $[-L, L]$, we have

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx.$$

(b) Show that for any odd function $f(x)$ defined on the interval $[-L, L]$, we have

$$\int_{-L}^{L} f(x)dx = 0.$$

Solution.
(a) Since $f(x)$ is even, we have $f(-x) = f(x)$ for all x in $[-L, L]$. Thus,

$$\int_{-L}^{0} f(x)dx = \int_{-L}^{0} f(-x)dx = -\int_{L}^{0} f(u)du = \int_{0}^{L} f(x)dx.$$

For this, it follows that

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx.$$

(b) Since $f(x)$ is odd, we have $f(-x) = -f(x)$ for all x in $[-L, L]$. Thus,

$$\int_{-L}^{0} f(x)dx = \int_{-L}^{0} [-f(-x)]dx = \int_{L}^{0} f(u)du = -\int_{0}^{L} f(x)dx.$$

Hence,

$$0 = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = \int_{-L}^{L} f(x)dx \blacksquare$$

Example 16.3
(a) Find the value of the integral $\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$ when f is even.
(b) Find the value of the integral $\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ when f is odd.

Solution.
(a) Since the function $\sin\left(\frac{n\pi}{L}x\right)$ is odd and f is even, we have that $f(x)\sin\left(\frac{n\pi}{L}x\right)$ is odd so that

$$\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = 0.$$

(b) Since the function $\cos\left(\frac{n\pi}{L}x\right)$ is even and f is odd, we have that $f(x)\cos\left(\frac{n\pi}{L}x\right)$ is odd so that

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = 0 \blacksquare$$

Even and Odd Extensions
Let $f : (0, L) \to \mathbb{R}$ be a piecewise smooth function. We define the **odd extension** of this function on the interval $-L < x < L$ by

$$f_{odd}(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0. \end{cases}$$

This function will be odd on the interval $(-L, L)$, and will be equal to $f(x)$ on the interval $(0, L)$. We can then further extend this function to the entire real line by defining it to be $2L$ periodic. Let $\overline{f}_{odd}$ denote this extension. We note that $\overline{f}_{odd}$ is an odd function and piecewise smooth so that by Theorem 15.2 it possesses a Fourier series expansion, and from the fact that it is odd all of the a_n's are zero. Moreover, in the interval $(0, L)$ we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right). \quad (16.1)$$

We call (16.1) the **Fourier sine series** of f.
The coefficients b_n are given by the formula

$$\begin{aligned} b_n =& \frac{1}{L}\int_{-L}^{L} \overline{f}_{odd} \sin\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_0^L \overline{f}_{odd} \sin\left(\frac{n\pi}{L}x\right)dx \\ =& \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

since $\overline{f}_{odd}\sin\left(\frac{n\pi}{L}x\right)$ is an even function.
Likewise, we can define the **even extension** of f on the interval $-L < x < L$ by

$$f_{even}(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0. \end{cases}$$

We can then further extend this function to the entire real line by defining it to be $2L$ periodic. Let $\overline{f}_{even}$ denote this extension. Again, we note that $\overline{f}_{even}$ is equal to the original function $f(x)$ on the interval upon which $f(x)$ is defined. Since $\overline{f}_{even}$ is piecewise smooth, by Theorem 15.2 it possesses a Fourier series expansion, and from the fact that it is even all of the b_n's are zero. Moreover, in the interval $(0, L)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right). \tag{16.2}$$

We call (16.2) the **Fourier cosine series** of f. The coefficients a_n are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx,\ n = 0, 1, 2, \cdots.$$

Example 16.4
Graph the odd and even extensions of the function $f(x) = x,\ 0 \le x \le 1$.

Solution.
We have $f_{odd}(x) = x$ for $-1 \le x \le 1$. The odd extension of f is shown in Figure 16.1(a). Likewise,

$$f_{even}(x) = \begin{cases} x & 0 \le x \le 1 \\ -x & -1 \le x < 0. \end{cases}$$

The even extension is shown in Figure 16.1(b) ■

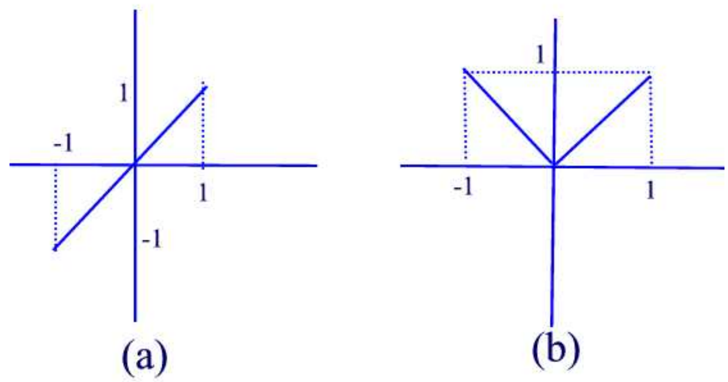

Figure 16.1

Example 16.5
Find the Fourier sine series of the function

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

Solution.
We have

$$b_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx dx\right].$$

Using integration by parts we find

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sin nx dx &= \left[-\frac{x}{n}\cos nx\right]_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}} \cos nx dx \\ &= -\frac{\pi \cos(n\pi/2)}{2n} + \frac{1}{n^2}[\sin nx]_0^{\frac{\pi}{2}} \\ &= -\frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2} \end{aligned}$$

while

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx dx &= \left[-\frac{(\pi - x)}{n}\cos nx\right]_{\frac{\pi}{2}}^{\pi} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi} \cos nx dx \\ &= \frac{\pi \cos(n\pi/2)}{2n} - \frac{1}{n^2}[\sin nx]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2}. \end{aligned}$$

Thus,

$$b_n = \frac{4\sin(n\pi/2)}{\pi n^2},$$

and the Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{\pi n^2} \sin nx = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi(2n-1)^2} \sin(2n-1)x \blacksquare$$

Practice Problems

Problem 16.1
Give an example of a function that is both even and odd.

Problem 16.2
Graph the odd and even extensions of the function $f(x) = 1,\ 0 \le x \le 1$.

Problem 16.3
Graph the odd and even extensions of the function $f(x) = L - x$ for $0 \le x \le L$.

Problem 16.4
Graph the odd and even extensions of the function $f(x) = 1 + x^2$ for $0 \le x \le L$.

Problem 16.5
Find the Fourier cosine series of the function

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

Problem 16.6
Find the Fourier cosine series of $f(x) = x$ on the interval $[0, \pi]$.

Problem 16.7
Find the Fourier sine series of $f(x) = 1$ on the interval $[0, \pi]$.

Problem 16.8
Find the Fourier sine series of $f(x) = \cos x$ on the interval $[0, \pi]$.

Problem 16.9
Find the Fourier cosine series of $f(x) = e^{2x}$ on the interval $[0, 1]$.

Problem 16.10
For the following functions on the interval $[0, L]$, find the coefficients b_n of the Fourier sine expansion.
(a) $f(x) = \sin\left(\frac{2\pi}{L}x\right)$.
(b) $f(x) = 1$
(c) $f(x) = \cos\left(\frac{\pi}{L}x\right)$.

Problem 16.11
For the following functions on the interval $[0, L]$, find the coefficients a_n of the Fourier cosine expansion.
(a) $f(x) = 5 + \cos\left(\frac{\pi}{L}x\right)$.
(b) $f(x) = x$
(c)

$$f(x) = \begin{cases} 1 & 0 < x \le \frac{L}{2} \\ 0 & \frac{L}{2} < x \le L. \end{cases}$$

Problem 16.12
Consider a function $f(x)$, defined on $0 \le x \le L$, which is even (symmetric) around $x = \frac{L}{2}$. Show that the even coefficients (n even) of the Fourier sine series are zero.

Problem 16.13
Consider a function $f(x)$, defined on $0 \le x \le L$, which is odd around $x = \frac{L}{2}$. Show that the even coefficients (n even) of the Fourier cosine series are zero.

Problem 16.14
The Fourier sine series of $f(x) = \cos\left(\frac{\pi x}{L}\right)$ for $0 \le x \le L$ is given by

$$\cos\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}$$

where

$$b_1 = 0, \quad b_n = \frac{2n}{(n^2 - 1)\pi}[1 + (-1)^n].$$

Using term-by-term integration, find the Fourier cosine series of $\sin\left(\frac{\pi x}{L}\right)$.

Problem 16.15
Consider the function

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 2 & 1 \le x < 2. \end{cases}$$

(a) Sketch the even extension of f.
(b) Find a_0 in the Fourier series for the even extension of f.
(c) Find a_n $(n = 1, 2, \cdots)$ in the Fourier series for the even extension of f.
(d) Find b_n in the Fourier series for the even extension of f.
(e) Write the Fourier series for the even extension of f.