15 An Introduction to Fourier Series

In this and the next section we will have a brief look to the subject of Fourier series. The point here is to do just enough to allow us to do some basic solutions to partial differential equations later in the book.
Motivation: In Calculus we have seen that certain functions may be represented as power series by means of the Taylor expansions. These functions must have infinitely many derivatives, and the series provide a good approximation only in some (often small) vicinity of a reference point.
Fourier series constructed of trigonometric rather than power functions, and can be used for functions not only not differentiable, but even discontinuous at some points. The main limitation of Fourier series is that the underlying function should be periodic.
Recall from calculus that a **function series** is a series where the summands are functions. Examples of function series include power series, Laurent series, Fourier series, etc.
Unlike series of numbers, there exist many types of convergence of series of functions, namely, pointwise, uniform, etc. We say that a series of functions $\sum_{n=1}^{\infty} f_n(x)$ **converges pointwise** to a function f if and only if the sequence of partial sums

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

converges pointwise to f. We write

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{n\to\infty} S_n(x) = f(x).$$

Example 15.1
Show that $\sum_{n=0}^{\infty} x^n$ converges pointwise to a function to be determined for all $-1 < x < 1$.

Solution.
The n^{th} term of the sequence of partial sums is given by

$$S_n(x) = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Since

$$\lim_{n\to\infty} x^{n+1} = 0, \quad -1 < x < 1,$$

the partial sums converge pointwise to the function $\frac{1}{1-x}$. Thus,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \blacksquare$$

Likewise, we say that a series of functions $\sum_{n=1}^{\infty} f_n(x)$ **converges uniformly** to a function f if and only if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges uniformly to f.
The following theorem provide a tool for uniform convergence of series of functions.

Theorem 15.1 *(Weierstrass M-test)*
Suppose that for each x in an interval I the series $\sum_{n=1}^{\infty} f_n(x)$ is well-defined. Suppose further that

$$|f_n(x)| \le M_n, \quad \forall x \in I.$$

If $\sum_{n=1}^{\infty} M_n$ (a scalar series) is convergent then the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent.

Example 15.2
Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is uniformly convergent.

Solution.
For all $x \in \mathbb{R}$, we have

$$\left|\frac{\sin(nx)}{n^2}\right| \le \frac{|\sin(nx)|}{n^2} \le \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent being a $p-$series with $p = 2 > 1$. Thus, by Weierstrass M-test the given series is uniformly convergent $\blacksquare$

In this section we introduce a type of series of functions known as **Fourier series**. They are given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right], \quad -L \le x \le L \qquad (15.1)$$

where a_n and b_n are called the **Fourier coefficients.** Note that we begin the series with $\frac{a_0}{2}$ as opposed to simply a_0 to simplify the coefficient formula

for a_n that we will derive later in this section.
The main questions we want to consider next are the questions of determining which functions can be represented by Fourier series and if so how to compute the coefficients a_n and b_n.
Before answering these questions, we look at some of the properties of Fourier series.

Periodicity Property
Recall that a function f is said to be **periodic** with period $T > 0$ if $f(x+T) = f(x)$ for all $x, x+T$ in the domain of f. The smallest value of T for which f is periodic is called the **fundamental period**. A graph of a $T-$periodic function is shown in Figure 15.1.

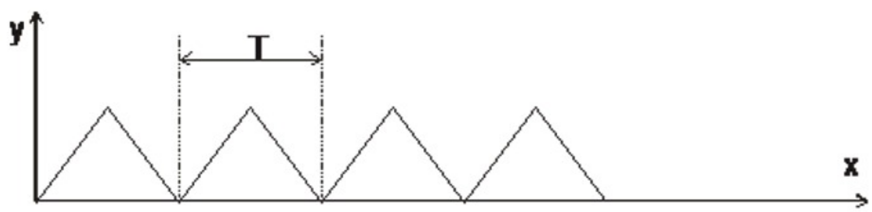

Figure 15.1

For a $T-$periodic function we have

$$f(x) = f(x+T) = f(x+2T) = \cdots .$$

Note that the definite integral of a $T-$periodic function is the same over any interval of length T. By Problem 15.1 below, if f and g are two periodic functions with common period T, then the product fg and an arbitrary linear combination $c_1 f + c_2 g$ are also periodic with period T. It is an easy exercise to show that the Fourier series (15.1) is periodic with fundamental period $2L$.

Orthogonality Property
Recall from Calculus that for each pair of vectors $\vec{u}$ and $\vec{v}$ we associate a scalar quantity $\vec{u} \cdot \vec{v}$ called the **dot product** of $\vec{u}$ and $\vec{v}$. We say that $\vec{u}$ and $\vec{v}$ are **orthogonal** if and only if $\vec{u} \cdot \vec{v} = 0$. We want to define a similar concept for functions.
Let f and g be two functions with domain the closed interval $[a, b]$. We define

a function that takes a pair of functions to a scalar. Symbolically, we write

$$< f,g >= \int_a^b f(x)g(x)dx.$$

We call $< f,g >$ the **inner product** of f and g. We say that f and g are **orthogonal** if and only if $< f,g >= 0$. A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.
Before we proceed any further into computations, we would like to remind the reader of the following two facts from calculus:
• If $f(x)$ is an odd function defined on the interval $[-L,L]$ then $\int_{-L}^{L} f(x)dx = 0$.
• If $f(x)$ is an even function defined on the interval $[-L,L]$ then $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.

Example 15.3
Show that the set $\left\{1, \cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right) : n \in \mathbb{N}\right\}$, where $m \neq n$, is mutually orthogonal in $[-L,L]$.

Solution.
Since the cosine function is even, we have

$$\int_{-L}^{L} 1\cdot\cos\left(\frac{n\pi}{L}x\right)dx = 2\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx = \frac{2L}{n\pi}\left[\sin\left(\frac{n\pi}{L}x\right)\right]_0^L = 0.$$

Since the sine function is odd, we have

$$\int_{-L}^{L} 1\cdot\sin\left(\frac{n\pi}{L}x\right)dx = 0.$$

Now, for $n \neq m$ we have

$$\begin{aligned}\int_{-L}^{L}\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx =& \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m+n)\pi}{L}x\right)+\cos\left(\frac{(m-n)\pi}{L}x\right)\right]dx\\ =&\frac{1}{2}\left[\frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi}{L}x\right)\right.\\ &+\left.\frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi}{L}x\right)\right]_{-L}^{L} = 0\end{aligned}$$

where we used the trigonometric identity

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)].$$

We can also show (see Problem 15.2):

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

and

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0 \blacksquare$$

The reason we care about these functions being orthogonal is because we will exploit this fact to develop a formula for the coefficients in our Fourier series.

Now, in order to answer the first question mentioned earlier, that is, which functions can be expressed as a Fourier series expansion, we need to introduce some mathematical concepts.
A function $f(x)$ is said to be **piecewise continuous** on $[a, b]$ if it is continuous in $[a, b]$ except possibly at finitely many points of discontinuity within the interval $[a, b]$, and at each point of discontinuity, the right- and left-handed limits of f exist. An example of a piecewise continuous function is the function

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ x^2 - x & 1 \le x \le 2. \end{cases}$$

We will say that f is **piecewise smooth** in $[a, b]$ if and only if $f(x)$ as well as its derivatives are piecewise continuous.
The following theorem, proven in more advanced books, ensures that a Fourier decomposition can be found for any function which is piecewise smooth.

Theorem 15.2
Let f be a $2L$-periodic function. If f is a piecewise smooth on $[-L, L]$ then for all points of discontinuity $x \in [-L, L]$ we have

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

where as for points of continuity $x \in [-L, L]$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right].$$

Remark 15.1
(1) Almost all functions occurring in practice are piecewise smooth functions.
(2) Given a piecewise smooth function f on $[-L, L]$. The above theorem applies to the periodic extension F of f where $F(x + 2nL) = f(x)$ $(n \in \mathbb{Z})$ and $F(x) = f(x)$ on $(-L, L)$. Note that if $f(-L) = f(L)$ then $F(x) = f(x)$ on $[-L, L]$. Otherwise, the end points of $f(x)$ may be jump discontinuities of $F(x)$.

Convergence Results of Fourier Series
We list few of the results regarding the convergence of Fourier series:
(1) The type of convergence in the above theorem is pointwise convergence.
(2) The convergence is uniform for a continuous function f on $[-L, L]$ such that $f(-L) = f(L)$.
(3) The convergence is uniform whenever $\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$ is convergent.
(4) If $f(x)$ is periodic, continuous, and has a piecewise continuous derivative, then the Fourier Series corresponding to f converges uniformly to $f(x)$ for the entire real line.
(5) The convergence is uniform on any closed interval that does not contain a point of discontinuity.

Euler-Fourier Formulas
Next, we will answer the second question mentioned earlier, that is, the question of finding formulas for the coefficients a_n and b_n. These formulas for a_n and b_n are called Euler-Fourier formulas which we derive next. We will assume that the series in (15.1) converges uniformly to $f(x)$ on the interval $[-L, L]$. That is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right], \quad -L \le x \le L. \tag{15.2}$$

Integrating both sides of (15.2) we obtain

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2}dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] dx.$$

Since the trigonometric series is assumed to be uniformly convergent, from Theorem 14.2, we can interchange the order of integration and summation to obtain

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2}dx + \sum_{n=1}^{\infty} \int_{-L}^{L} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] dx.$$

But

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)dx = \frac{L}{n\pi} \sin\left(\frac{n\pi}{L}x\right)\Big]_{-L}^{L} = 0$$

and likewise

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)dx = -\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right)\Big]_{-L}^{L} = 0.$$

Thus,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx.$$

To find the other Fourier coefficients, we recall the results of Problems 15.2 - 15.3 below.

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = 0, \ \forall m, n.$$

Now, to find the formula for the Fourier coefficients a_m for $m > 0$, we multiply both sides of (15.2) by $\cos\left(\frac{m\pi}{L}x\right)$ and integrate from $-L$ to L to otbain

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right) = \int_{-L}^{L} \frac{a_0}{2}\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx \right.$$
$$\left. + b_n \int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx\right].$$

Hence,

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx = a_m L$$

and therefore

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx.$$

Likewise, we can show that

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx.$$

Example 15.4
Find the Fourier series expansion of

$$f(x) = \begin{cases} 0, & x \le 0 \\ x, & x > 0 \end{cases}$$

on the interval $[-\pi, \pi]$.

Solution.
We have

$$\begin{aligned} a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{\pi}{2} \\ a_n =& \frac{1}{\pi}\int_0^{\pi} x\cos nxdx = \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{(-1)^n - 1}{\pi n^2} \\ b_n =& \frac{1}{\pi}\int_0^{\pi} x\sin nxdx = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{(-1)^{n+1}}{n}. \end{aligned}$$

Hence,

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right] \quad -\pi < x < \pi \;\blacksquare$$

Example 15.5
Apply Theorem 15.2 to the function in Example 15.4.

Solution.
Let F be a periodic extension of f of period 2π. See Figure 152.

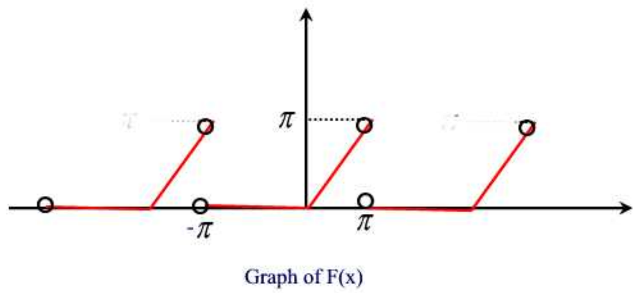

Figure 152

Thus, $f(x) = F(x)$ on the interval $(-\pi, \pi)$. Note that for $x = \pi$, the Fourier series coverges to

$$\frac{F(\pi^-) + F(\pi^+)}{2} = \frac{\pi}{2}.$$

Similar result for $x = -\pi$. Clearly, F is a piecewise smooth function so that by the previous thereom we can write

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right] = \begin{cases} \frac{\pi}{2}, & \text{if } x = -\pi \\ f(x), & \text{if } -\pi < x < \pi \\ \frac{\pi}{2}, & \text{if } x = \pi. \end{cases}$$

Taking $x = \pi$ we have the identity

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}(-1)^n = \frac{\pi}{2}$$

which can be simplified to

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This provides a method for computing an approximate value of π ■

Remark 15.2
An example of a function that does not have a Fourier series representation is the function $f(x) = \frac{1}{x^2}$ on $[-L, L]$. For example, the coefficient a_0 for this function does not exist. Thus, not every function can be written as a Fourier series expansion.

The final topic of discussion here is the topic of differentiation and integration of Fourier series. In particular we want to know if we can differentiate a Fourier series term by term and have the result be the Fourier series of the derivative of the function. Likewise we want to know if we can integrate a Fourier series term by term and arrive at the Fourier series of the integral of the function. Answers to these questions are provided next.

Theorem 15.3
A Fourier series of a *piecewise smooth* function f can always be integrated term by term and the result is a convergent infinite series that always converges to $\int_{-L}^{L} f(x)dx$ even if the original series has jumps.

Theorem 15.4
A Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f'(x)$ is *piecewise smooth.* The result of the differentiation is the Fourier series of $f'(x)$.

Practice Problems

Problem 15.1
Let f and g be two functions with common domain D and common period T. Show that
(a) fg is periodic of period T.
(b) $c_1f + c_2g$ is periodic of period T, where c_1 and c_2 are real numbers.

Problem 15.2
Show that for $m \neq n$ we have
(a) $\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0$ and
(b) $\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0$.

Problem 15.3
Compute the following integrals:
(a) $\int_{-L}^{L} \cos^2\left(\frac{n\pi}{L}x\right) dx$.
(b) $\int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right) dx$.
(c) $\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx$.

Problem 15.4
Find the Fourier coefficients of

$$f(x) = \begin{cases} -\pi, & -\pi \leq x < 0 \\ \pi, & 0 < x < \pi \\ 0, & x = 0, \pi \end{cases}$$

on the interval $[-\pi, \pi]$.

Problem 15.5
Find the Fourier series of $f(x) = x^2 - \frac{1}{2}$ on the interval $[-1, 1]$.

Problem 15.6
Find the Fourier series of the function

$$f(x) = \begin{cases} -1, & -2\pi < x < -\pi \\ 0, & -\pi < x < \pi \\ 1, & \pi < x < 2\pi. \end{cases}$$

Problem 15.7
Find the Fourier series of the function

$$f(x) = \begin{cases} 1+x, & -2 \le x \le 0 \\ 1-x, & 0 < x \le 2. \end{cases}$$

Problem 15.8
Show that $f(x) = \frac{1}{x}$ is not piecewise continuous on $[-1, 1]$.

Problem 15.9
Assume that $f(x)$ is continuous and has period $2L$. Prove that

$$\int_{-L}^{L} f(x)dx = \int_{-L+a}^{L+a} f(x)dx$$

is independent of $a \in \mathbb{R}$. In particular, it does not matter over which interval the Fourier coefficients are computed as long as the interval length is $2L$. [Remark: This result is also true for piecewise continuous functions].

Problem 15.10
Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 2 & 1 \le x < 3 \end{cases}$$

and extended periodically with period 3 to $\mathbb{R}$ so that $f(x+3) = f(x)$ for all x.
(i) Find the Fourier series of $f(x)$.
(ii) Discuss its limit: In particular, does the Fourier series converge pointwise or uniformly to its limit, and what is this limit?
(iii) Plot the graphs of $f(x)$ and its extension $F(x)$ on the interval $[0, 3]$.

Problem 15.11
For the following functions $f(x)$ on the interval $-L < x < L$, determine the coefficients $a_n,\ n = 0, 1, 2, \cdots$ and $b_n, n \in \mathbb{N}$ of the Fourier series expansion.
(a) $f(x) = 1$.
(b) $f(x) = 2 + \sin\left(\frac{\pi x}{L}\right)$.
(c) $f(x) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0. \end{cases}$
(d) $f(x) = x$.

Problem 15.12
Let $f(t)$ be the function with period 2π defined as

$$f(t) = \begin{cases} 2 & \text{if } 0 \le x \le \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \le 2\pi \end{cases}$$

$f(t)$ has a Fourier series and that series is equal to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \ 0 < x < 2\pi.$$

Find $\frac{a_3}{b_3}$.

Problem 15.13
Let $f(x) = x^3$ on $[-\pi, \pi]$, extended periodically to all of $\mathbb{R}$. Find the Fourier coefficients $a_n, \ n = 1, 2, 3, \cdots$.

Problem 15.14
Let $f(x)$ be the square wave function

$$f(x) = \begin{cases} -\pi & -\pi \le x < 0 \\ \pi & 0 \le x \le \pi \end{cases}$$

extended periodically to all of $\mathbb{R}$. To what value does the Fourier series of $f(x)$ converge when $x = 0$?

Problem 15.15
(a) Find the Fourier series of

$$f(x) = \begin{cases} 1 & -\pi \le x < 0 \\ 2 & 0 \le x \le \pi \end{cases}$$

extended periodically to all of $\mathbb{R}$. Simplify your coefficients as much as possible.
(b) Use (a) to evaluate the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$. Hint: Evaluate the Fourier series at $x = \frac{\pi}{2}$.