14 Sequences of Functions: Pointwise and Uniform Convergence

In the next section, we will be constructing solutions to PDEs involving infinite sums of sines and cosines. These infinite sums or series are called **Fourier series.** Fourier series are examples of series of functions. Convergence of series of functions is defined in terms of convergence of a sequence of functions. In this section we study the two types of convergence of sequences of functions.

Recall that a sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is said to **converge** to a number L if and only if for every given $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that for all $n \geq N$ we have$|a_n - L| < \epsilon$.

What is the analogue concept of convergence when the terms of the sequence are variables? Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$ consider a function $f_n : D \to \mathbb{R}$. Thus, we obtain a sequence of functions $\{f_n\}_{n=1}^{\infty}$. For such a sequence, there are two types of convergenve that we consider in this section: pointwise convergence and uniform convergence.

We say that $\{f_n\}_{n=1}^{\infty}$ **converges pointwise** in D to a function $f : D \to \mathbb{R}$ if and only if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in D$. Equivalently, for a given $x \in D$ and $\epsilon > 0$ there is a positive integer $N = N(x, \epsilon)$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$. It is important to note that N is a function of both x and ϵ.

Example 14.1

Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function $f(x) = 0$ for all $x \geq 0$.

Solution.

For all $x \geq 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1 + n^2x^2} = 0 = f(x) \blacksquare$$

Example 14.2

For each positive integer n let $f_n : (0, \infty) \to (0, \infty)$ be given by $f_n(x) = nx$. Show that $\{f_n\}_{n=1}^{\infty}$ does not converge pointwise in $D = (0, \infty)$.

Solution.
This follows from the fact that $\lim_{n\to\infty} nx = \infty$ for all $x \in D$ ■

One of the weaknesses of this type of convergence is that it does not preserve some of the properties of the base functions $\{f_n\}_{n=1}^{\infty}$. For example, if each f_n is continuous then the pointwise limit function need not be continuous. (See Problem 14.1) A stronger type of convergence which preserves most of the properties of the base functions is the uniform convergence which we define next.
Let D be a subset of $\mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on D. We say that $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** on D to a function $f : D \to \mathbb{R}$ if and only if for all $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.
This definition says that the integer N depends only on the given ϵ (in contrast to pointwise convergence where N depends on both x and ϵ) so that for $n \geq N$, the graph of $f_n(x)$ is bounded above by the graph of $f(x) + \epsilon$ and below by the graph of $f(x) - \epsilon$ as shown in Figure 14.1.

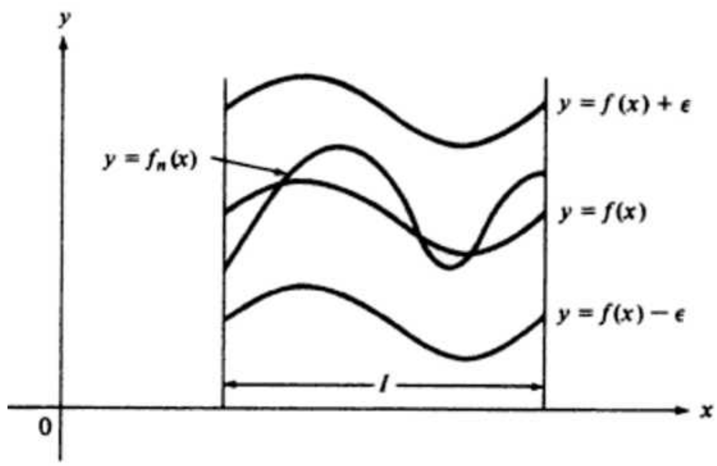

Figure 14.1

Example 14.3
For each positive integer n let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$. Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the zero function.

Solution.
Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for

$n \geq N$ we have

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

for all $x \in [0, 1]$ ■

Clearly, uniform convergence implies pointwise convergence to the same limit function. However, the converse is not true in general. Thus, one way to show that a sequence of functions does not converge uniformly is to show that it does not converge pointwise.

Example 14.4
Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. By Example 14.1, this sequence converges pointwise to $f(x) = 0$. Let $\epsilon = \frac{1}{3}$. Show that there is no positive integer N with the property $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \geq 0$. Hence, the given sequence does not converge uniformly to $f(x)$.

Solution.
For any positive integer N and for $n \geq N$ we have

$$\left|f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = \frac{1}{2} > \epsilon \ ■$$

Problem 14.1 shows a sequence of continuous functions converging pointwise to a discontinuous function. That is, pointwise convergence does not preserve the property of continuity. One of the interesting features of uniform convergence is that it preserves continuity as shown in the next example.

Example 14.5
Suppose that for each $n \geq 1$ the function $f_n : D \to \mathbb{R}$ is continuous in D. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. Let $a \in D$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$.
(b) Show that there is a $\delta > 0$ such that for all $|x - a| < \delta$ we have $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$.
(c) Using (a) and (b) show that for $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Hence, f is continuous in D since a was arbitrary.

Solution.
(a) This follows from the definition of uniform convergence.
(b) This follows from the fact that f_N is continuous at $a \in D$.
(c) For $|x-a| < \delta$ we have $|f(x) - f(a)| = |f(a) - f_N(a) + f_N(a) - f_N(x) + f_N(x) - f(x)| \leq |f_N(a) - f(a)| + |f_N(a) - f_N(x)| + |f_N(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ ■

From this example, we can write

$$\lim_{x \to a} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to a} f_n(x).$$

Indeed,

$$\begin{aligned}\lim_{x \to a} \lim_{n \to \infty} f_n(x) =& \lim_{x \to a} f(x) \\ =& f(a) = \lim_{n \to \infty} f_n(a) \\ =& \lim_{n \to \infty} \lim_{x \to a} f_n(x).\end{aligned}$$

Does pointwise convergence allow the interchange of limits and integration? The answer is no as shown in the next example.

Example 14.6
The sequence of function $f_n : (0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$ converges pointwise to the zero function. Show that

$$\lim_{n \to \infty} \int_1^\infty f_n(x) dx \neq \int_1^\infty \lim_{n \to \infty} f_n(x) dx.$$

Solution.
We have

$$\int_1^\infty \frac{x}{n} dx = \left. \frac{x^2}{2n} \right|_1^\infty = \infty.$$

Hence,

$$\lim_{n \to \infty} \int_1^\infty f_n(x) dx = \infty$$

whereas

$$\int_1^\infty \lim_{n \to \infty} f_n(x) dx = 0 \ ■$$

Contrary to pointwise convergence, uniform convergence preserves integration. That is, if $\{f_n\}_{n=1}^\infty$ converges uniformly to f on a closed interval $[a,b]$ then

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

Theorem 14.1
Suppose that $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to $f : [a,b] \to \mathbb{R}$. Then

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Proof.
From Example 14.5, we have that f is continuous and hence integrable. Let $\epsilon > 0$ be given. By uniform convergence, we can find a positive integer N such that $|f_n(x) - f(x) < \frac{\epsilon}{b-a}$ for all x in $[a,b]$ and $n \geq N$. Thus, for $n \geq N$, we have

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \leq \int_a^b |f_n(x) - f(x)|dx < \epsilon.$$

This completes the proof of the theorem ■

Now, what about differentiability? Again, pointwise convergence fails in general to conserve the differentiability property. See Problem 14.1. Does uniform convergence preserve differentiability? The answer is still no as shown in the next example.

Example 14.7
Consider the family of functions $f_n : [-1,1]$ given by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.
(a) Show that f_n is differentiable for each $n \geq 1$.
(b) Show that for all $x \in [-1,1]$ we have

$$|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}}$$

where $f(x) = |x|$. Hint: Note that $\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2} \geq \frac{1}{\sqrt{n}}$.
(c) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for $n \geq N$ we have

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in [-1, 1].$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the non-differentiable function $f(x) = |x|$.

Solution.
(a) f_n is the composition of two differentiable functions so it is differentiable with derivative

$$f_n'(x) = x\left[x^2 + \frac{1}{n}\right]^{-\frac{1}{2}}.$$

(b) We have

$$|f_n(x) - f(x)| = \left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| = \left|\frac{(\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2})(\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2})}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}\right|$$

$$= \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}$$

$$\leq \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}}.$$

(c) Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ we can find a positive integer N such that for all $n \geq N$ we have $\frac{1}{\sqrt{n}} < \epsilon$. Now the answer to the question follows from this and part (b) ■

Even when uniform convergence occurs, the process of interchanging limits and differentiation may fail as shown in the next example.

Example 14.8
Consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{\sin nx}{n}$.
(a) Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function $f(x) = 0$.
(b) Note that $\{f_n\}_{n=1}^{\infty}$ and f are differentiable functions. Show that

$$\lim_{n\to\infty} f_n'(x) \neq f'(x) = \left[\lim_{n\to\infty} f_n(x)\right]'.$$

That is, one cannot, in general, interchange limits and derivatives.

Solution.
(a) Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$ we have

$$|f_n(x) - f(x)| = \left|\frac{\sin nx}{n}\right| \leq \frac{1}{n} < \epsilon$$

and this is true for all $x \in \mathbb{R}$. Hence, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function $f(x) = 0$.
(b) We have $\lim_{n\to\infty} f_n'(\pi) = \lim_{n\to\infty} \cos n\pi = \lim_{n\to\infty}(-1)^n$ which does not converge. However, $f'(\pi) = 0$ ■

Pointwise convergence was not enough to preserve differentiability, and neither was uniform convergence by itself. Even with uniform convergence the process of interchanging limits with derivatives is not true in general. However, if we combine pointwise convergence with uniform convergence we can indeed preserve differentiability and also switch the limit process with the process of differentiation.

Theorem 14.2
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions on $[a, b]$ that converges pointwise to some function f defined on $[a, b]$. If $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to a function g, then the function f is differentiable with derivative equals to g. Thus,

$$\lim_{n\to\infty} f_n'(x) = g(x) = f'(x) = \left[\lim_{n\to\infty} f_n(x)\right]'.$$

Proof.
First, note that the function g is continuous in $[a, b]$ since uniform convergence preserves continuity. Let c be an arbitrary point in $[a, b]$. Then

$$\int_c^x f_n'(t)dt = f_n(x) - f_n(c),\ x \in [a, b].$$

Taking the limit of both sides and using the facts that f_n' converges uniformly to g and f_n converges pointwise to f, we can write

$$\int_c^x g(t)dt = f(x) - f(c).$$

Taking the derivative of both sides of the last equation yields $g(x) = f'(x)$ ■

Finally, we conclude this section with the following important result that is useful in testing uniform convergence.

Theorem 14.3
Consider a sequence $f_n : D \to \mathbb{R}$. Then this sequence converges uniformly to $f : D \to \mathbb{R}$ if and only if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

Proof.
Suppose that f_n converges uniformly to f. Let $\epsilon > 0$ be given. Then there is a positive integer N such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $n \geq N$ and all $x \in D$. Thus, for $n \geq N$, we have

$$\sup\{|f_n(x) - f(x)| : x \in D\} \leq \frac{\epsilon}{2} < \epsilon.$$

This shows that

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

Conversely, suppose that

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

Let $\epsilon > 0$ be given. Then there is a positive interger N such that

$$\sup\{|f_n(x) - f(x)| : x \in D\} < \epsilon$$

for all $n \geq N$. But this implies that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in D$. Hence, f_n converges uniformly to f in D ■

Example 14.9
Show that the sequence defined by $f_n(x) = \frac{\cos x}{n}$ converges uniformly to the zero function.

Solution.
We have
$$0 \leq \sup\{|\frac{\cos x}{n}| : x \in \mathbb{R}\} \leq \frac{1}{n}.$$
Now apply the squeeze rule[1] for sequences we find that
$$\lim_{n\to\infty} \sup\{|\frac{\cos x}{n}| : x \in \mathbb{R}\} = 0$$
which implies that the given sequence converges uniformly to the zero function on $\mathbb{R}$ ■

[1] If $a_n \leq b_n \leq c_n$ for all $n \geq N$ and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

Practice Problems

Problem 14.1
Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(a) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.
(b) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f. Hint: Suppose otherwise. Let $\epsilon = 0.5$ and get a contradiction by using a point $(0.5)^{\frac{1}{N}} < x < 1$.

Problem 14.2
Consider the sequence of functions

$$f_n(x) = \frac{nx + x^2}{n^2}$$

defined for all x in $\mathbb{R}$. Show that this sequence converges pointwise to a function f to be determined.

Problem 14.3
Consider the sequence of functions

$$f_n(x) = \frac{\sin(nx + 3)}{\sqrt{n+1}}$$

defined for all x in $\mathbb{R}$. Show that this sequence converges pointwise to a function f to be determined.

Problem 14.4
Consider the sequence of functions defined by $f_n(x) = n^2 x^n$ for all $0 \le x \le 1$. Show that this sequence does not converge pointwise to any function.

Problem 14.5
Consider the sequence of functions defined by $f_n(x) = (\cos x)^n$ for all $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Show that this sequence converges pointwise to a noncontinuous function to be determined.

Problem 14.6
Consider the sequence of functions $f_n(x) = x - \frac{x^n}{n}$ defined on $[0, 1)$.
(a) Does $\{f_n\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.
(b) Does $\{f_n'\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.

Problem 14.7
Let $f_n(x) = \frac{x^n}{1+x^n}$ for $x \in [0, 2]$.
(a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ on $[0, 2]$.
(b) Does $f_n \to f$ uniformly on $[0, 2]$?

Problem 14.8
For each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$.
(a) Show that $f_n \to \frac{1}{2}$ uniformly.
(b) Find $\lim_{n\to\infty} \int_2^7 f_n(x)dx$.

Problem 14.9
Show that the sequence defined by $f_n(x) = (\cos x)^n$ does not converge uniformly on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Problem 14.10
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions such that

$$\sup\{|f_n(x)| : 2 \le x \le 5\} \le \frac{2^n}{1+4^n}.$$

(a) Show that this sequence converges uniformly to a function f to be found.
(b) What is the value of the limit $\lim_{n\to\infty} \int_2^5 f_n(x)dx$?