13 Parabolic Type: The Heat Equation in One-Dimensional Space

In this section, We will look at a model for describing the distribution of temperature in a solid material as a function of time and space. More specifically, we will derive the heat equation that models the flow of heat in a rod that is insulated everywhere except at the two ends.

Before we begin our discussion of the mathematics of the heat equation, we must first determine what is meant by the term **heat**? Heat is type of energy known as **thermal energy**. Heat travels in waves like other forms of energy, and can change the matter it touches. It can heat it up and cause chemical reactions like burning to occur.

Heat can be released through a chemical reaction (such as the nuclear reactions that make the Sun "burn") or can be trapped for a limited time by insulators. It is often released along with other kinds of energy such as light waves or sound waves. For example, a burning candle releases light and heat waves. On the other hand, an explosion releases light, heat, and sound waves. The most common units of heat are BTU (British Thermal Unit), Calorie and Joule.

Consider now a thin rod made of homogeneous heat conducting material of uniform density ρ and constant cross section A, wrapped along the $x-$axis from $x = 0$ to $x = L$ as shown in Figure 13.1.

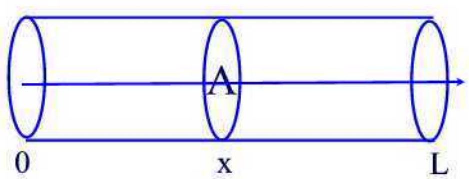

Figure 13.1

Assume the heat flows only in the $x-$direction, with the lateral sides well insulated, and the only way heat can enter or leave the rod is at either end. Since our rod is thin, the temperature of the rod can be considered constant on any cross section and so depends on the horizontal position along the $x-$axis and we can hence consider the rod to be a one spatial dimensional rod. We will also assume that heat energy in any piece of the rod is conserved. That is, the heat gained at one end is equal to the heat lost at the other end.

Let $u(x,t)$ be the temperature of the cross section at the point x and the time t. Consider a portion U of the rod from x to $x+\Delta x$ of length Δx as shown in Figure 13.2.

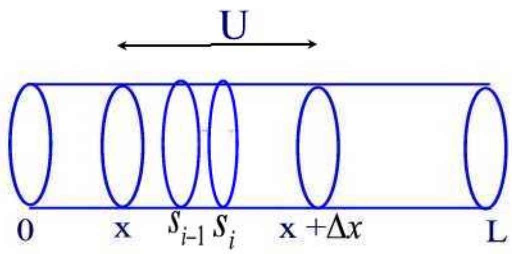

Figure 13.2

Divide the interval $[x, x+\Delta x]$ into n sub-intervals each of length Δs using the partition points $x = s_0 < s_1 < \cdots < s_n = x+\Delta x$. Consider the portion U_i of U of height Δs. The portion U_i is assumed to be thin so that the temperature is constant throughout the volume. From the theory of heat conduction, the quantity of heat Q_i in U_i at time t is given by

$$Q_i = cm_i u(s_{i-1}, t) = c\rho u(s_{i-1}, t)\Delta V_i$$

where m_i is the mass of U_i, ΔV_i is the volume of U_i and c is the **specific heat**, that is, the amount of heat that it takes to raise one unit of mass of the material by one unit of temperature.
But U_i is a cylinder of height Δs and area of base A so that $\Delta V_i = A\Delta s$. Hence,

$$Q_i = c\rho A u(s_{i-1}, t)\Delta s.$$

The quantity of heat in the portion U is given by

$$Q(t) = \lim_{n\to\infty}\sum_{i=1}^{n} Q_i = \lim_{n\to\infty}\sum_{i=1}^{n} c\rho A u(s_{i-1}, t)\Delta s = \int_x^{x+\Delta x} c\rho A u(s,t)ds.$$

By differentiation, the change in heat with respect to time is

$$\frac{dQ}{dt} = \int_x^{x+\Delta x} c\rho A u_t(s,t)ds.$$

Assuming that u is continuously differentiable, we can apply the mean value theorem for integrals and find $x \leq \xi \leq x + \Delta x$ such that

$$\int_x^{x+\Delta x} u_t(s,t)ds = \Delta x u_t(\xi, t).$$

Thus, the rate of change of heat in U is given by

$$\frac{dQ}{dt} = c\rho A \Delta x u_t(\xi, t).$$

Now, Fourier law of heat transfer says that the rate of heat transfer through any cross section is proportional to the area A and the negative gradient of the temperature normal to the cross section, i.e., $-KAu_x(x,t)$. Note that if the temperature increases as x increases (i.e., the temperature is hotter to the right), $u_x > 0$ so that the heat flows to the left. This explains the minus sign in the formula for Fourier law. Hence, according to this law heat is transferred from areas of high temperature to areas of low temperature.
Now, the rate of heat flowing in U through the cross section at x is $-KAu_x(x,t)$ and the rate of heat flowing out of U through the cross section at $x + \Delta x$ is $-KAu_x(x + \Delta x, t)$, where K is the **thermal conductivity**[1] of the rod.
Now, the conservation of energy law states

$$\text{rate of change of heat in } U = \text{rate of heat flowing in} - \text{rate of heat flowing out}$$

or mathematically written as,

$$c\rho A \Delta x u_t(\xi, t) = -KAu_x(x,t) + KAu_x(x + \Delta x, t)$$

or

$$c\rho A \Delta x u_t(\xi, t) = KA[u_x(x + \Delta x, t) - u_x(x,t)].$$

Dividing this last equation by $cA\rho\Delta x$ and letting $\Delta x \to 0$ we obtain

$$u_t(x,t) = ku_{xx}(x,t) \tag{13.1}$$

where $k = \frac{K}{c\rho}$ is called the **diffusivity constant.**
Equation (13.1) is the one dimensional heat equation which is second order,

[1] It is a property of material to conduct heat. Heat transfer is slow in materials with small thermal conductivity and fast in materials with large thermal conductivity.

linear, homogeneous, and of parabolic type.
The non-homogeneous heat equation

$$u_t = ku_{xx} + f(x)$$

is known as the heat equation with an external **heat source** $f(x)$. An example of an external heat source is the heat generated from a candle placed under the bar.
The function

$$E(t) = \int_0^L u(x,t)dx$$

is called the **total thermal energy**[2] at time t of the entire rod.

Example 13.1
The two ends of a homogeneous rod of length L are insulated. There is a constant source of thermal energy $q_0 \neq 0$ and the temperature is initially $u(x,0) = f(x)$.
(a) Write the equation and the boundary conditions for this model.
(b) Calculate the total thermal energy of the entire rod.

Solution.
(a) The model is given by the PDE

$$u_t(x,t) = ku_{xx} + q_0$$

with boundary conditions

$$u_x(0,t) = u_x(L,t) = 0.$$

(b) First note that

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)dx = &\int_0^L u_t(x,t)dx = \int_0^L ku_{xx}dx + \int_0^L q_0 dx\\ &= ku_x|_0^L + q_0L = q_0L\end{aligned}$$

since $u_x(0,t) = u_x(L,t) = 0$. Integrating with respect to t we find

$$E(t) = q_0Lt + C.$$

[2] The total internal energy in the rod generated by the rod's temperature.

But $C = E(0) = \int_0^L u(x,0)dx = \int_0^L f(x)dx$. Hence, the total thermal energy is given by

$$E(t) = \int_0^L f(x)dx + q_0 Lt \blacksquare$$

Initial Boundary Value Problems

In order to solve the heat equation we must give the problem some initial conditions. If you recall from the theory of ODE, the number of conditions required for solving initial value problems always matched the highest order of the derivative in the equation.

In partial differential equations the same idea holds except now we have to pay attention to the variable we are differentiating with respect to as well. So, for the heat equation we have got a first order time derivative and so we will need one initial condition and a second order spatial derivative and so we will need two boundary conditions.

For the initial condition, we define the temperature of every point along the rod at time $t = 0$ by

$$u(x,0) = f(x)$$

where f is a given (prescribed) function of x. This function is known as the **initial temperature distribution.**

The boundary conditions will tell us something about what the temperature is doing at the ends of the bar. The conditions are given by

$$u(0,t) = T_0 \text{ and } u(L,t) = T_L.$$

and they are called as the **Dirichlet conditions.** In this case, the general form of the heat equation initial boundary value problem is to find $u(x,t)$ satisfying

$$\begin{aligned} u_t(x,t) =& k u_{xx}(x,t), \quad 0 \le x \le L, \ t > 0 \\ u(x,0) =& f(x), \quad 0 \le x \le L \\ u(0,t) =& T_0, u(L,t) = T_L, \quad t > 0. \end{aligned}$$

In the case of insulated endpoints, i.e., there is no heat flow out of them, we use the boundary conditions

$$u_x(0,t) = u_x(L,t) = 0.$$

These conditions are examples of what is known as **Neumann boundary** conditions. In this case, the general form of the heat equation initial boundary value problem is to find $u(x,t)$ satisfying

$$\begin{aligned} u_t(x,t) =& k u_{xx}(x,t), \quad 0 \leq x \leq L, \; t > 0 \\ u(x,0) =& f(x), \quad 0 \leq x \leq L \\ u_x(0,t) =& u_x(L,t) = 0, \quad t > 0. \end{aligned}$$

Practice Problems

Problem 13.1
Show that if $u(x,t)$ and $v(x,t)$ satisfy equation (13.1) then $\alpha u + \beta v$ is also a solution to (13.1), where α and β are constants.

Problem 13.2
Show that any linear time independent function $u(x,t) = ax + b$ is a solution to equation (13.1).

Problem 13.3
Find a linear time independent solution u to (13.1) that satisfies $u(0,t) = T_0$ and $u(L,T) = T_L$.

Problem 13.4
Show that to solve (13.1) with the boundary conditions $u(0,t) = T_0$ and $u(L,t) = T_L$ it suffices to solve (13.1) with the **homogeneous boundary conditions** $u(0,t) = u(L,t) = 0$.

Problem 13.5
Find a solution to (13.1) that satisfies the conditions $u(x,0) = u(0,t) = u(L,t) = 0$.

Problem 13.6
Let (I) denote equation (13.1) together with intial condition $u(x,0) = f(x)$, where f is not the zero function, and the homogeneous boundary conditions $u(0,t) = u(L,t) = 0$. Suppose a nontrivial solution to (I) can be written in the form $u(x,t) = X(x)T(t)$. Show that X and T satisfy the ODE

$$X'' - \tfrac{\lambda}{k}X = 0 \text{ and } T' - \lambda T = 0$$

for some constant λ.

Problem 13.7
Consider again the solution $u(x,t) = X(x)T(t)$. Clearly, $T(t) = T(0)e^{\lambda t}$. Suppose that $\lambda > 0$.
(a) Show that $X(x) = Ae^{x\sqrt{\alpha}} + Be^{-x\sqrt{\alpha}}$, where $\alpha = \frac{\lambda}{k}$ and A and B are arbitrary constants.
(b) Show that A and B satisfy the two equations $A + B = 0$ and $A(e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}) = 0$.
(c) Show that $A = 0$ leads to a contradiction.
(d) Using (b) and (c) show that $e^{L\sqrt{\alpha}} = e^{-L\sqrt{\alpha}}$. Show that this equality leads to a contradiction. We conclude that $\lambda < 0$.

Problem 13.8
Consider the results of the previous exercise.
(a) Show that $X(x) = c_1 \cos \beta x + c_2 \sin \beta x$ where $\beta = \sqrt{\frac{-\lambda}{k}}$.
(b) Show that $\lambda = \lambda_n = -\frac{kn^2\pi^2}{L^2}$, where n is an integer.

Problem 13.9
Show that $u(x,t) = \sum_{i=1}^n u_i(x,t)$, where $u_i(x,t) = c_i e^{-\frac{ki^2\pi^2}{L^2}t} \sin\left(\frac{i\pi}{L}x\right)$ satisfies (13.1) and the homogeneous boundary conditions.

Problem 13.10
Suppose that a wire is stretched between 0 and a. Describe the boundary conditions for the temperature $u(x,t)$ when
(i) the left end is kept at 0 degrees and the right end is kept at 100 degrees; and
(ii) when both ends are insulated.

Problem 13.11
Let $u_t = u_{xx}$ for $0 < x < \pi$ and $t > 0$ with boundary conditions $u(0,t) = 0 = u(\pi,t)$ and initial condition $u(x,0) = \sin x$. Let $E(t) = \int_0^\pi (u_t^2 + u_x^2)dx$. Show that $E'(t) < 0$.

Problem 13.12
Suppose

$$u_t = u_{xx} + 4, \;\; u_x(0,t) = 5, \;\; u_x(L,t) = 6, \;\; u(x,0) = f(x).$$

Calculate the total thermal energy of the one-dimensional rod (as a function of time).

Problem 13.13
Consider the heat equation

$$u_t = ku_{xx}$$

for $x \in (0,1)$ and $t > 0$, with boundary conditions $u(0,t) = 2$ and $u(1,t) = 3$ for $t > 0$ and initial condition $u(x,0) = x$ for $x \in (0,1)$. A function $v(x)$ that satisfies the equation $v''(x) = 0$, with conditions $v(0) = 2$ and $v(1) = 3$ is called a **steady-state solution.** That is, the steady-state solutions of the heat equation are those solutions that don't depend on time. Find $v(x)$.

Problem 13.14
Consider the equation for the one-dimensional rod of length L with given heat energy source:

$$u_t = u_{xx} + q(x).$$

Assume that the initial temperature distribution is given by $u(x,0) = f(x)$. Find the equilibrium (steady state) temperature distribution in the following cases.
(a) $q(x) = 0, u(0) = 0, u(L) = T$.
(b) $q(x) = 0, u_x(0) = 0, u(L) = T$.
(c) $q(x) = 0, u(0) = T, u_x(L) = \alpha$.

Problem 13.15
Consider the equation for the one-dimensional rod of length L with insulated ends:

$$u_t = ku_{xx}, \quad u_x(0,t) = u_x(L,t) = 0.$$

(a) Give the expression for the total thermal energy of the rod.
(b) Show using the equation and the boundary conditions that the total thermal energy is constant.

Problem 13.16
Suppose

$$u_t = u_{xx} + x, \quad u(x,0) = f(x),\ u_x(0,t) = \beta,\ u_x(L,t) = 7.$$

(a) Calculate the total thermal energy of the one-dimensional rod (as a function of time).
(b) From part (a) find the value of β for which a steady-state solution exist.
(c) For the above value of β find the steady state solution.