12 Hyperbolic Type: The Wave equation

The wave equation has many physical applications from sound waves in air to magnetic waves in the Sun's atmosphere. However, the simplest systems to visualize and describe are waves on a stretched flexible string.
A flexible homogeneous string of length L and constant mass density ρ (i.e., mass per unit length) is stretched horizontally along the $x-$axis with its left end placed at $x = 0$ and its right end placed at $x = L$. From the left end (and at time $t = 0$) we slightly shake the string and we notice a small vibrations propagate through the string. We make the following physical assumptions:
(a) the string does not furnish any resistance to bending (i.e., perfectly elastic);
(b) the (pulling) tension force on the string is the dominant force and all other forces acting on the string are negligible (no external forces are applied to the string, the damping forces (resistance) and gravitational forces are negligible);
(c) clearly a point on the string moves up and down along a curve but since the horizontal displacement is small compared to the vertical displacement, we will assume that each point of the string moves only vertically. Thus, the horizontal component of the tension force must be constant.
We denote the vertical displacement from the $x-$axis of the string by $u(x, t)$ which is a function of position x and time t. That is, $u(x, t)$ is the vertical displacement from the equilibrium at position x and time t. Our aim is to find an equation that is satisfied by $u(x, t)$.
A displacement of a tiny piece of the string between points P and Q is shown in Figure 12.1,

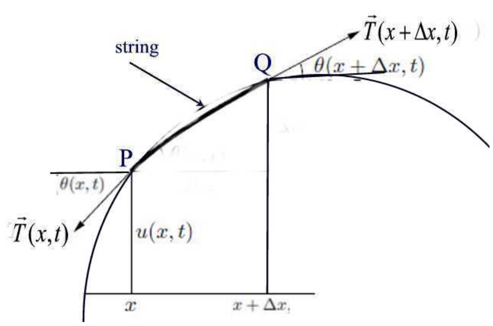

Figure 12.1

where
(i) $\theta(x,t)$ is the angle between $\vec{T}(x,t)$ and $\vec{i}$ at x and time t; for small vibrations, we have $\theta \approx 0$;
(ii) $\vec{T}(x,t)$ is the (pulling) tension force in the string at position x and time t pulling to the left and $\vec{T}(x+\Delta x,t)$ the tension force at position $x+\Delta x$ and t pulling the string to the right.
By (c) above, we have

$$||\vec{T}(x,t)|| \cos[\theta(x,t)] = ||\vec{T}(x+\Delta x,t)|| \cos[\theta(x+\Delta x,t)] = T.$$

Now, at P the vertical component of the tension force is $-||\vec{T}(x,t)|| \sin[\theta(x,t)]$ (the minus sign occurs due to the component at P is pointing downward) whereas at Q the vertical component is $||\vec{T}(x+\Delta x,t)|| \sin[\theta(x+\Delta x,t)]$. Then Newton's Law of motion

$$\text{mass} \times \text{acceleration} = \text{net applied forces}$$

gives

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = ||\vec{T}(x+\Delta x,t)|| \sin[\theta(x+\Delta x,t)] - ||\vec{T}(x,t)|| \sin[\theta(x,t)].$$

Next, dividing through by T, we obtain

$$\begin{aligned}
\frac{\rho}{T}\Delta x \frac{\partial^2 u}{\partial t^2} =& \frac{||\vec{T}(x+\Delta x,t)|| \sin[\theta(x+\Delta x,t)]}{||\vec{T}(x+\Delta x,t)|| \cos[\theta(x+\Delta x,t)]} - \frac{||\vec{T}(x,t)|| \sin[\theta(x,t)]}{||\vec{T}(x,t)|| \cos[\theta(x,t)]} \\
=& \tan[\theta(x+\Delta x,t)] - \tan[\theta(x,t)] \\
=& u_x(x+\Delta x,t) - u_x(x,t).
\end{aligned}$$

Dividing by Δx and letting $\Delta x \to 0$ we obtain

$$\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = u_{xx}(x,t).$$

which can be written as

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \tag{12.1}$$

where $c^2 = \frac{T}{\rho}$. We call c the **wave speed.**

General Solution of (12.1): D'Alembert Approach
By using the change of variables $v = x + ct$ and $w = x - ct$, we find

$$\begin{aligned}
u_t =& cu_v - cu_w \\
u_{tt} =& c^2 u_{vv} - 2c^2 u_{wv} + c^2 u_{ww} \\
u_x =& u_v + u_w \\
u_{xx} =& u_{vv} + 2u_{vw} + u_{ww}
\end{aligned}$$

Substituting into Equation (12.1), we find $u_{vw} = 0$ and solving this equation we find $u_v = F(v)$ and $u(v, w) = f(v) + g(w)$ where $f(v) = \int F(v)dv$.
Finally, using the fact that $v = x + ct$ and $w = x - ct$; we get d'Alembert's solution to the one-dimensional wave equation:

$$u(x, t) = f(x + ct) + g(x - ct) \tag{12.2}$$

where f and g are arbitrary differentiable functions.
The function $f(x + ct)$ represents waves that are moving to the left at a constant speed c and the function $g(x - ct)$ represents waves that are moving to the right at the same speed c.
The function $u(x, t)$ in (12.2) involves two arbitrary functions that are determined (normally) by two initial conditions.

Example 12.1
Find the solution to the Cauchy problem

$$\begin{aligned}
u_{tt} =& c^2 u_{xx} \\
u(x, 0) =& v(x) \\
u_t(x, 0) =& w(x).
\end{aligned}$$

Solution.
The condition $u(x, 0)$ is the initial position whereas $u_t(x, 0)$ is the initial velocity. We have

$$u(x, 0) = f(x) + g(x) = v(x)$$

and

$$u_t(x, 0) = -cf'(x) + cg'(x) = w(x)$$

which implies that

$$f(x) - g(x) = -\frac{1}{c}W(x) = -\frac{1}{c}\int_0^x w(s)ds.$$

Therefore,

$$f(x) = \frac{1}{2}(v(x) - \frac{1}{c}W(x))$$

and

$$g(x) = \frac{1}{2}(v(x) + \frac{1}{c}W(x)).$$

Finally,

$$\begin{aligned} u(x,t) =& \frac{1}{2}[v(x-ct) + v(x+ct) + \frac{1}{c}(W(x+ct) - W(x-ct))] \\ =& \frac{1}{2}[v(x-ct) + v(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} w(s)ds] \blacksquare \end{aligned}$$

Practice Problems

Problem 12.1
Show that if $v(x,t)$ and $w(x,t)$ satisfy equation (12.1) then $\alpha v + \beta w$ is also a solution to (12.1), where α and β are constants.

Problem 12.2
Show that any linear time independent function $u(x,t) = ax + b$ is a solution to equation (12.1).

Problem 12.3
Find a solution to (12.1) that satisfies the homogeneous conditions $u(x,0) = u(0,t) = u(L,t) = 0$.

Problem 12.4
Solve the initial value problem

$$\begin{aligned} u_{tt} =& 9u_{xx} \\ u(x,0) =& \cos x \\ u_t(x,0) =& 0. \end{aligned}$$

Problem 12.5
Solve the initial value problem

$$\begin{aligned} u_{tt} =& u_{xx} \\ u(x,0) =& \frac{1}{1+x^2} \\ u_t(x,0) =& 0. \end{aligned}$$

Problem 12.6
Solve the initial value problem

$$\begin{aligned} u_{tt} =& 4u_{xx} \\ u(x,0) =& 1 \\ u_t(x,0) =& \cos(2\pi x). \end{aligned}$$

Problem 12.7
Solve the initial value problem

$$\begin{aligned} u_{tt} =& 25u_{xx} \\ u(x,0) =& v(x) \\ u_t(x,0) =& 0 \end{aligned}$$

where

$$v(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Problem 12.8
Solve the initial value problem

$$\begin{aligned} u_{tt} =& c^2u_{xx} \\ u(x,0) =& e^{-x^2} \\ u_t(x,0) =& \cos^2 x. \end{aligned}$$

Problem 12.9
Prove that the wave equation, $u_{tt} = c^2u_{xx}$ satisfies the following properties, which are known as invariance properties. If $u(x,t)$ is a solution, then
(i) Any translate, $u(x-y,t)$ where y is a fixed constant, is also a solution.
(ii) Any derivative, say $u_x(x,t)$, is also a solution.
(iii) Any dilation, $u(ax,at)$, is a solution, for any fixed constant a.

Problem 12.10
Find $v(r)$ if $u(r,t) = \frac{v(r)}{r}\cos nt$ is a solution to the PDE

$$u_{rr} + \frac{2}{r}u_r = u_{tt}.$$

Problem 12.11
Find the solution of the wave equation on the real line $(-\infty < x < +\infty)$ with the initial conditions

$$u(x,0) = e^x, \quad u_t(x,0) = \sin x.$$

Problem 12.12
The total energy of the string (the sum of the kinetic and potential energies) is defined as

$$E(t) = \frac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)dx.$$

(a) Using the wave equation derive the equation of conservation of energy

$$\frac{dE(t)}{dt} = c^2(u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)).$$

(b) Assuming fixed ends boundary conditions, that is the ends of the string are fixed so that $u(0,t) = u(L,t) = 0$, for all $t > 0$, show that the energy is constant.
(c) Assuming free ends boundary conditions for both $x = 0$ and $x = L$, that is both $u(0,t)$ and $u(L,t)$ vary with t, show that the energy is constant.

Problem 12.13
For a wave equation with damping

$$u_{tt} - c^2 u_{xx} + du_t = 0, \quad d > 0, \ 0 < x < L$$

with the fixed ends boundary conditions show that the total energy decreases.

Problem 12.14
(a) Verify that for any twice differentiable $R(x)$ the function

$$u(x,t) = R(x - ct)$$

is a solution of the wave equation $u_{tt} = c^2 u_{xx}$. Such solutions are called traveling waves.
(b) Show that the potential and kinetic energies (see Exercise 12.12) are equal for the traveling wave solution in (a).

Problem 12.15
Find the solution of the Cauchy wave equation

$$u_{tt} = 4u_{xx}$$

$$u(x,0) = x^2, \quad u_t(x,0) = \sin 2x.$$

Simplify your answer as much as possible.