Second Order Linear Partial Differential Equations

In this chapter we consider the three fundamental second order linear partial differential equations of parabolic, hyperbolic, and elliptic type. These types arise in many applications such as the wave equation, the heat equation and the Laplace's equation. We will study the solvability of each of these equations.

11 Second Order PDEs in Two Variables

In this section we will briefly review second order partial differential equations.
A **second order partial differential equation in the variables** x and y is an equation of the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0. \tag{11.1}$$

If Equation (11.1) can be written in the form

$$A(x, y, u, u_x, u_y)u_{xx}+B(x, y, u, u_x, u_y)u_{xy}+C(x, y, u, u_x, u_y)u_{yy} = D(x, y, u, u_x, u_y) \tag{11.2}$$

then we say that the equation is **quasi-linear**.
If Equation (11.1) can be written in the form

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = D(x, y, u, u_x, u_y) \tag{11.3}$$

then we say that the equation is **semi-linear**.
If Equation (11.1) can be written in the form

$$A(x, y)u_{xx}+B(x, y)u_{xy}+C(x, y)u_{yy}+D(x, y)u_x+E(x, y)u_y+F(x, y)u = G(x, y) \tag{11.4}$$

then we say that the equation is **linear**.

A linear equation is said to be **homogeneous** when $G(x, y) \equiv 0$ and **non-homogeneous** otherwise.
Equation (11.4) resembles the general equation of a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which is classified as either parabolic, hyperbolic, or elliptic based on the sign of the **discriminant** $B^2 - 4AC$. We do the same for a second order linear partial differential equation:
• **Hyperbolic:** This occurs if $B^2 - 4AC > 0$ at a given point in the domain of u.
• **Parabolic:** This occurs if $B^2 - 4AC = 0$ at a given point in the domain of u.
• **Elliptic:** This occurs if $B^2 - 4AC < 0$ at a given point in the domain of u.

Example 11.1
Determine whether the equation $u_{xx} + xu_{yy} = 0$ is hyperbolic, parabolic or elliptic.

Solution.
Here we are given $A = 1, B = 0$, and $C = x$. Since $B^2 - 4AC = -4x$, the given equation is hyperbolic if $x < 0$, parabolic if $x = 0$ and elliptic if $x > 0$ ■

Second order partial differential equations arise in many areas of scientific applications. In what follows we list some of the well-known models that are of great interest:
1. The *heat equation* in one-dimensional space is given by

$$u_t = ku_{xx}$$

where k is a constant.
2. The *wave equation* in one-dimensional space is given by

$$u_{tt} = c^2 u_{xx}$$

where c is a constant.
3. The *Laplace equation* is given by

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Practice Problems

Problem 11.1
Classify each of the following equation as hyperbolic, parabolic, or elliptic:
(a) Wave propagation: $u_{tt} = c^2 u_{xx},\ c > 0.$
(b) Heat conduction: $u_t = cu_{xx},\ c > 0.$
(c) Laplace's equation: $\Delta u = u_{xx} + u_{yy} = 0.$

Problem 11.2
Classify the following linear scalar PDE with constant coefficents as hyperbolic, parabolic or elliptic.
(a) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0.$
(b) $u_{xx} - 4u_{xy} + 4u_{yy} + 3u_x + 4u = 0.$
(c) $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0.$

Problem 11.3
Find the region(s) in the $xy-$plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch these regions.

Problem 11.4
Show that $u(x,t) = \cos x \sin t$ is a solution to the problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= 0 \\ u_t(x,0) &= \cos x \\ u_x(0,t) &= 0 \end{aligned}$$

for all $x, t > 0.$

Problem 11.5
Classify each of the following PDE as linear, quasilinear, semi-linear, or nonlinear.
(a) $u_t + uu_x = uu_{xx}$
(b) $xu_{tt} + tu_{yy} + u^3 u_x^2 = t + 1$
(c) $u_{tt} = c^2 u_{xx}$
(d) $u_{tt}^2 + u_x = 0.$

Problem 11.6
Show that, for all $(x, y) \neq (0, 0)$, $u(x, y) = \ln(x^2 + y^2)$ is a solution of

$$u_{xx} + u_{yy} = 0,$$

and that, for all $(x, y, z) \neq (0, 0, 0)$, $u(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a solution of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Problem 11.7
Consider the eigenvalue problem

$$u_{xx} = \lambda u, \quad 0 < x < L$$

$$u_x(0) = k_0 u(0)$$

$$u_x(L) = -k_L u(L)$$

with Robin boundary conditions, where k_0 and k_L are given positive numbers and $u = u(x)$. Can this system have a nontrivial solution $u \not\equiv 0$ for $\lambda > 0$? Hint: Multiply the first equation by u and integrate over $x \in [0, L]$.

Problem 11.8
Show that $u(x, y) = f(x)g(y)$, where f and g are arbitrary differentiable functions, is a solution to the PDE

$$uu_{xy} = u_x u_y.$$

Problem 11.9
Show that for any $n \in \mathbb{N}$, the function $u_n(x, y) = \sin nx \sinh ny$ is a solution to the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Problem 11.10
Solve

$$u_{xy} = xy.$$

Problem 11.11
Classify each of the following second-oder PDEs according to whether they are hyperbolic, parabolic, or elliptic:
(a) $2u_{xx} - 4u_{xy} + 7u_{yy} - u = 0$.
(b) $u_{xx} - 2\cos x u_{xy} - \sin^2 x u_{yy} = 0$.
(c) $yu_{xx} + 2(x - 1)u_{xy} - (y + 2)u_{yy} = 0$.

Problem 11.12
Let $c > 0$. By computing u_x, u_{xx}, u_t, and u_{tt} show that

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

is a solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

where f is twice differentiable function and g is a differentiable function. Then compute and simplify $u(x,0)$ and $u_t(x,0)$.

Problem 11.13
Consider the second-order PDE

$$yu_{xx} + u_{xy} - x^2 u_{yy} - u_x - u = 0.$$

Determine the region D in $\mathbb{R}^2$, if such a region exists, that makes this PDE: (a) hyperbolic, (b) parabolic, (c) elliptic.

Problem 11.14
Consider the second-order hyperbolic PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

Use the change of variables $v(x,y) = y - 3x$ and $w(x,y) = x + y$ to solve the given equation.

Problem 11.15
Solve the Cauchy problem

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

$$u(x,2x) = 1,\ u_x(x,2x) = x.$$