10 The Cauchy Problem for First Order Quasilinear Equations

When solving a partial differential equation, it is seldom the case that one tries to study the properties of the general solution of such equations. In general, one deals with those partial differential equations whose solutions satisfy certain supplementary conditions. In the case of a first order partial differential equation, we determine the particular solution by formulating an initial value porblem also known as a **Cauchy problem.**

In this section, we discuss the Cauchy problem for the first order quasilinear partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{10.1}$$

Recall that the initial value problem of a first order ordinary differential equation asks for a solution of the equation which has a given value at a given point in $\mathbb{R}$. The Cauchy problem for the PDE (10.1) asks for a solution of (10.1) which has given values on a given curve in $\mathbb{R}^2$. A precise statement of the problem is given next.

Initial Value Problem or Cauchy Problem

Let C be a given curve in $\mathbb{R}^2$ defined parametrically by the equations

$$x = x_0(t), \quad y = y_0(t)$$

where x_0, y_0 are continuously differentiable functions on some interval I. Let $u_0(t)$ be a given continuously differentiable function on I. The Cauchy problem for (10.1) asks for a continuously differentiable function $u = u(x,y)$ defined in a domain $\Omega \subset \mathbb{R}^2$ containing the curve C and such that:

(1) $u = u(x,y)$ is a solution of (10.1) in Ω.

(2) On the curve C, u equals the given function $u_0(t)$, i.e.

$$u(x_0(t), y_0(t)) = u_0(t), \ t \in I. \tag{10.2}$$

We call C the **initial curve** of the problem, $u_0(t)$ the **initial data**, and (10.2) the **initial condition** or **Cauchy data** of the problem. See Figure 10.1.

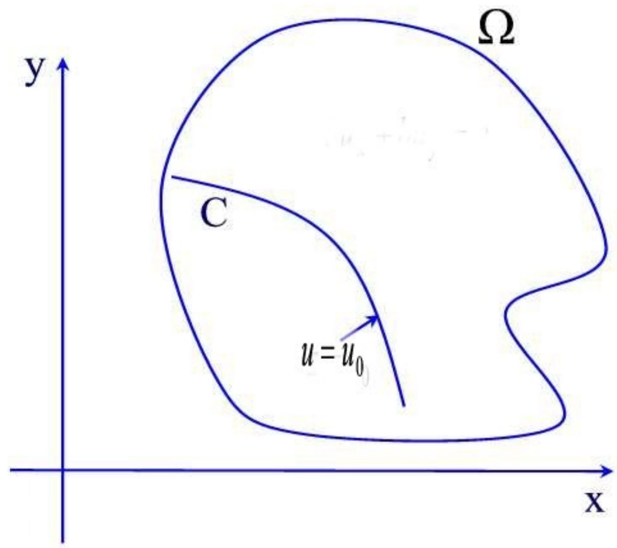

Figure 10.1

If we view a solution $u = u(x, y)$ of (10.1) as an integral surface of (10.1), we can give a simple geometrical statement of the problem: Find a solution surface of (10.1) containing the curve Γ described parametrically by the equations

$$\Gamma : x = x_0(t),\ y = y_0(t),\ u = u_0(t), \quad t \in I.$$

Note that the projection of this curve in the $xy-$plane is just the curve C. The following theorem asserts that under certain conditions the Cauchy problem (10.1) - (10.2) has a unique solution.

Theorem 10.1
Suppose that $x_0(t), y_0(t)$, and $u_0(t)$ are continuously differentiable functions of t in an interval I, and that a, b, and c are functions of x, y, and u with continuous first order partial derivatives with respect to their argument in some domain D of $(x, y, u)-$space containing the initial curve

$$\Gamma : x = x_0(t),\ y = y_0(t),\ u = u_0(t)$$

where $t \in I$. If $(x_0(t), y_0(t), u_0(t))$ is a point on Γ that satisfies the condition

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) \neq 0 \qquad (10.3)$$

then by continuity this relation holds in a neighborhood U of $(x_0(t), y_0(t))$ so that Γ is nowhere characteristic in U. In this case, there exists a unique solution $u = u(x, y)$ of (10.1) in U such that the initial condition (10.2) is satisfied for every point on C contained in U. See Figure 10.2. That is, there is a unique integral surface of (10.1) that contains Γ in a neighborhood of $(x_0(t), y_0(t))$.

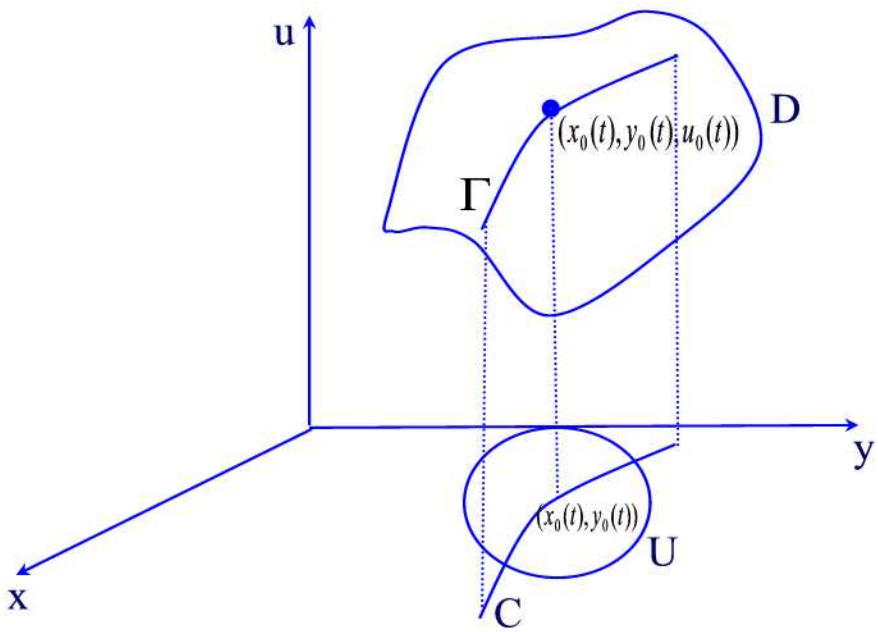

Figure 10.2

We construct the desired solution using the method of characteristics as follows: Pick a point $(x_0(t), y_0(t), u_0(t)) \in \Gamma$. Using this as the initial value we solve the system of ODEs consisting of the characteristic equations in parametric form

$$\begin{aligned} \frac{dx}{ds} &= a \\ \frac{dy}{ds} &= b \\ \frac{du}{ds} &= c \end{aligned}$$

satisfying the initial condition

$$(x(0), y(0), u(0)) = (x_0(t), y_0(t), u_0(t)).$$

The solution depends on the parameter s so it consists of a triples of functions

$$x = x(s,t), \quad y = y(s,t), \quad u = u(s,t). \tag{10.4}$$

This system represents the parametric representation of the integral surface of the problem in which the curve Γ corresponds to $s = 0$. The solution u is recovered by solving the first two equations in (10.4) for

$$t = t(x,y), \quad s = s(x,y)$$

and substituting these into the third equation to obtain $u(x,y) = u(s(x,y), t(x,y))$. We illustrate this process next.

Example 10.1
Solve the Cauchy problem

$$\begin{aligned} u_x + u_y =& 1 \\ u(x,0) =& f(x). \end{aligned}$$

Solution.
The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$\Gamma : x_0(t) = t, \; y_0(t) = 0, \; u_0(t) = f(t).$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = -1 \neq 0$$

so by the above theorem the given Cauchy problem has a unique solution. To find this solution, we solve the system of ODEs

$$\begin{aligned} \frac{dx}{ds} =& 1 \\ \frac{dy}{ds} =& 1 \\ \frac{du}{ds} =& 1. \end{aligned}$$

Solving this system we find

$$x(s,t) = s + \alpha(t), \quad y(s,t) = s + \beta(t), \quad u(s) = s + \gamma(t).$$

But $x(0,t) = t$ so that $\alpha(t) = t$. Similarly, $y(0,t) = 0$ so that $\beta(t) = 0$ and $u(0,t) = f(t)$ implies $\gamma(t) = f(t)$. Hence, the unique solution is given parametrically by the equations

$$x(s,t) = t + s, \quad y(s,t) = s, \quad u(s,t) = s + f(t).$$

Solving the first two equations for s and t we find

$$s = y, \quad t = x - y$$

and substituting these into the third equation we find

$$u(x,y) = y + f(x - y).$$

Alternative Computation
We can apply the results of the previous section to find the unique solution. If we solve the characteristic equations in non-parametric form

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{1}$$

we find $x - y = c_1$ and $u - x = c_2$. Thus, the general solution of the PDE is given by $u = x + F(x - y)$. Using the Cauchy data $u(x,0) = f(x)$ we find $f(x) = x + F(x)$ which implies that $F(x) = f(x) - x$. Hence, the unique solution is given by

$$u(x,y) = x + f(x - y) - (x - y) = y + f(x - y) \blacksquare$$

If condition (10.3) is not satisfied then C is a characteristic curve. If the curve Γ satisfies the characteristic equations than the problem has infinitely many solutions. To see this, pick an arbitrary point $P_0 = (x_0, y_0, u_0)$ on Γ. Pick a new initial curve Γ' passing through P_0 which is not tangent to Γ at P_0. In this case, condition (10.3) is satisfied and the new Cauchy problem has a unique solution. Since there are infinitely many ways of selecting Γ', we obtain infinitely many solutions. We illustrate this case in the next example.

Example 10.2
Solve the Cauchy problem

$$\begin{aligned} u_x + u_y =& 1 \\ u(x,x) =& x. \end{aligned}$$

Solution.
The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$\Gamma : x_0(t) = t,\ \ y_0(t) = t,\ \ u_0(t) = t.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = 0.$$

As in Example 10.1, the general solution of the PDE is $u(x,y) = y + f(x - y)$ where f is an arbitrary differentiable function. Using the Cauchy data $u(x,x) = x$ we find $f(0) = 0$. Thus, the solution is given by

$$u(x,y) = y + f(x - y)$$

where f is an arbitrary function such that $f(0) = 0$. There are infinitely many choices for f. Hence, the problem has infinitely many solutions. Note that Γ satisfies the characteristic equations ■

If condition (10.3) is not satisfied and if Γ does not satisfy the characteristic equations then it can be shown that the Cauchy problem has no solutions. We illustrate this case next.

Example 10.3
Solve the Cauchy problem

$$\begin{aligned} u_x + u_y =& 1 \\ u(x,x) =& 1. \end{aligned}$$

Solution.
The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$\Gamma : x_0(t) = t,\ \ y_0(t) = t,\ \ u_0(t) = 1.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = 0.$$

Solving the characteristic equations in parametric form we find

$$x(s,t) = s + \alpha(t), \quad y(s,t) = s + \beta(t), \quad u(s,t) = s + \gamma(t).$$

Clearly, Γ does not satisfy the characteristic equations. Now, the general solution to the PDE is given by $u = y + f(x - y)$. Using the Cauchy data $u(x, x) = 1$ we find $f(0) = 1 - x$, which is not possible since the LHS is a fixed number whereas the RHS is a variable expression. Hence, the problem has no solutions ■

Example 10.4
Solve the Cauchy problem

$$\begin{aligned} u_x - u_y =& 1 \\ u(x, 0) =& x^2. \end{aligned} \qquad (10.5)$$

Solution.
The initial curve is given parametrically by

$$\Gamma : \; x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = t^2.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = 1 \neq 0$$

so the Cauchy problem has a unique solution.
The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}.$$

Using the first two fractions we find $x + y = c_1$. Using the first and the third fractions we find $u - x = c_2$. Thus, the general solution can be represented by

$$u = x + f(x + y)$$

where f is an arbitrary differentiable function. Using the Cauchy data $u(x, 0) = x^2$ we find $x^2 - x = f(x)$. Hence, the unique solution is given by

$$u = x + (x + y)^2 - (x + y) = (x + y)^2 - y \; ■$$

Example 10.5
Solve the initial value problem

$$u_t + uu_x = x, \quad u(x,0) = 1.$$

Solution.
The initial curve is given parametrically by

$$\Gamma : \ x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = 1.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = -1 \neq 0$$

so the Cauchy problem has a unique solution.
The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{x}.$$

Since

$$\frac{dt}{1} = \frac{d(x+u)}{x+u}$$

we find that $(x+u)e^{-t} = c_1$. Now, using the last two fractions we find $u^2 - x^2 = c_2$. Hence, the general solution is given by

$$f((x+u)e^{-t}, u^2 - x^2) = 0$$

where f is an arbitrary differentiable function. Using the Cauchy data we find $c_1 = 1 + x$ and $c_2 = 1 - x^2 = 2(1+x) - (1+x)^2 = 2c_1 - c_1^2$. Thus,

$$u^2 - x^2 = 2(x+u)e^{-t} - (x+u)^2 e^{-2t}$$

or

$$u - x = 2e^{-t} - (x+u)e^{-2t}.$$

This can be reduced further as follows: $u + ue^{-2t} = x + 2e^{-t} - xe^{-2t} = 2e^{-t} + x(1 - e^{-2t}) \Longrightarrow u = \frac{2e^{-t}}{1+e^{-2t}} + x\frac{1-e^{-2t}}{1+e^{-2t}} = \text{sech}(t) + x\text{tanh}(t)$ ■

Example 10.6
Solve the initial value problem

$$uu_x + u_y = 1$$

with the initial curve

$$\Gamma : x_0(t) = 2t^2, \quad y_0(t) = 2t, \quad u_0(s) = 0, \quad t > 0.$$

Solution.
We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = -4t \neq 0, \ t > 0$$

so the Cauchy problem has a unique solution.
The characteristic equations in parametric form are given by the system of ODEs

$$\begin{aligned}\frac{dx}{ds} =& u\\ \frac{dy}{ds} =& 1\\ \frac{du}{ds} =& 1.\end{aligned}$$

Thus, the solution of this system depends on two parameters s and t. Solving the last two equations we find

$$y(s,t) = s + \beta(t), \quad u(s,t) = s + \gamma(t).$$

Solving the first equation with u being replaced by $s + \gamma(t)$ we find

$$x(s,t) = \frac{1}{2}s^2 + \gamma(t)s + \alpha(t).$$

Using the initial conditions

$$x(0,t) = 2t^2, \ y(0,t) = 2t, \ u(0,t) = 0$$

we find

$$x(s,t) = \frac{1}{2}s^2 + 2t^2, \quad y(s,t) = s + 2t, \quad u(s,t) = s.$$

Eliminating s and t we find

$$(u-y)^2+u^2=2x.$$

Solving this quadratic equation in u to find

$$2u=y\pm(4x-y^2)^{\frac{1}{2}}.$$

The solution surface satisfying $u=0$ on $y^2=2x$ is given by

$$2u=y-(4x-y^2)^{\frac{1}{2}}.$$

This represents a solution surface only when $y^2<4x$. The solution does not exist for $y^2>4x$ ■

Practice Problems

Problem 10.1
Solve

$$(y-u)u_x + (u-x)u_y = x - y$$

with the condition $u(x, \frac{1}{x}) = 0$.

Problem 10.2
Solve the linear equation

$$yu_x + xu_y = u$$

with the Cauchy data $u(x, 0) = x^3$.

Problem 10.3
Solve

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

with the Cauchy data $u(x, -x) = 1$.

Problem 10.4
Solve

$$xu_x + yu_y = xe^{-u}$$

with the Cauchy data $u(x, x^2) = 0$.

Problem 10.5
Solve the initial value problem

$$xu_x + u_y = 0, \quad u(x, 0) = f(x)$$

using the characteristic equations in parametric form.

Problem 10.6
Solve the initial value problem

$$u_t + au_x = 0, \quad u(x, 0) = f(x).$$

Problem 10.7
Solve the initial value problem

$$au_x + u_y = u^2, \quad u(x, 0) = \cos x$$

Problem 10.8
Solve the initial value problem

$$u_x + xu_y = u, \quad u(1, y) = h(y).$$

Problem 10.9
Solve the initial value problem

$$uu_x + u_y = 0, \quad u(x, 0) = f(x).$$

Problem 10.10
Solve the initial value problem

$$\sqrt{1 - x^2}u_x + u_y = 0, \quad u(0, y) = y.$$

Problem 10.11
Consider

$$xu_x + 2yu_y = 0.$$

(i) Find and sketch the characteristics.
(ii) Find the solution with $u(1, y) = e^y$.
(iii) What happens if you try to find the solution satisfying either $u(0, y) = g(y)$ or $u(x, 0) = h(x)$ for given functions g and h?
(iv) Explain, using your picture of the characteristics, what goes wrong at $(x, y) = (0, 0)$.

Problem 10.12
Solve the equation $u_x + u_y = u$ subject to the condition $u(x, 0) = \cos x$.

Problem 10.13
(a) Find the general solution of the equation

$$u_x + yu_y = u.$$

(b) Find the solution satisfying the Cauchy data $u(x, 3e^x) = 2$.
(c) Find the solution satisfying the Cauchy data $u(x, e^x) = e^x$.

Problem 10.14
Solve the Cauchy problem

$$u_x + 4u_y = x(u + 1)$$

$$u(x, 5x) = 1.$$

Problem 10.15
Solve the Cauchy problem

$$u_x - u_y = u$$
$$u(x, -x) = \sin x.$$

Problem 10.16
(a) Find the characteristics of the equation

$$yu_x + xu_y = 0.$$

(b) Sketch some of the characteristics.
(c) Find the solution satisfying the boundary condition $u(0, y) = e^{-y^2}$.
(d) In which region of the plane is the solution uniquely determined?

Problem 10.17
Consider the equation $u_x + yu_y = 0$. Is there a solution satisfying the extra condition
(a) $u(x, 0) = 1$
(b) $u(x, 0) = x$?
If yes, give a formula; if no, explain why.