The Basics of the Theory of Partial Differential Equation

Many fields in engineering and the physical sciences require the study of ODEs and PDEs. Some of these fields include acoustics, aerodynamics, elasticity, electrodynamics, fluid dynamics, geophysics (seismic wave propagation), heat transfer, meteorology, oceanography, optics, petroleum engineering, plasma physics (ionized liquids and gases), quantum mechanics.
So the study of partial differential equations is of great importance to the above mentioned fields. The purpose of this chapter is to introduce the reader to the basic terms of partial differential equations.

1 Basic Concepts

The goal of this section is to introduce the reader to the basic concepts and notations that will be used in the remainder of this book.
We start this section by reviewing the concept of partial derivatives and the chain rule of functions in two variables.
Let $u(x,y)$ be a function of the independent variables x and y. The **first derivative of u with respect to x** is defined by

$$u_x(x,y) = \frac{\partial u}{\partial x}(x,y) = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

provided that the limit exists.
Likewise, the **first derivative of u with respect to y** is defined by

$$u_y(x,y) = \frac{\partial u}{\partial y}(x,y) = \lim_{h \to 0} \frac{u(x,y+h) - u(x,y)}{h}$$

provided that the limit exists.
We can define higher order derivatives such as

$$u_{xx}(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) = \lim_{h \to 0} \frac{u_x(x+h,y) - u_x(x,y)}{h}$$

$$u_{yy}(x,y) = \frac{\partial^2 u}{\partial y^2}(x,y) = \lim_{h \to 0} \frac{u_y(x,y+h) - u_y(x,y)}{h}$$

$$u_{xy}(x,y) = \frac{\partial^2 u}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \lim_{h \to 0} \frac{u_x(x,y+h) - u_x(x,y)}{h}$$

and

$$u_{yx}(x,y) = \frac{\partial^2 u}{\partial x \partial y}(x,y) = \lim_{h \to 0} \frac{u_y(x+h,y) - u_y(x,y)}{h}$$

provided that the limits exist.[1]
An important formula of differentiation is the so-called **chain rule.** If $u = u(x,y)$, where $x = x(s,t)$ and $y = y(s,t)$, then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

Likewise,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

[1] If u_{xy} and u_{yx} are continuous then $u_{xy}(x,y) = u_{yx}(x,y)$.

Example 1.1
Compute the partial derivatives indicated:
(a) $\frac{\partial}{\partial y}(y^2 \sin xy)$
(b) $\frac{\partial^2}{\partial x^2}[e^{x+y}]^2$

Solution.
(a) We have $\frac{\partial}{\partial y}(y^2 \sin xy) = \sin xy \frac{\partial}{\partial y}(y^2) + y^2 \frac{\partial}{\partial y}(\sin xy) = 2y \sin xy + xy^2 \cos xy$.
(b) We have $\frac{\partial}{\partial x}[e^{x+y}]^2 = \frac{\partial}{\partial x} e^{2(x+y)} = 2e^{2(x+y)}$. Thus, $\frac{\partial^2}{\partial x^2}[e^{x+y}]^2 = \frac{\partial}{\partial x} 2e^{2(x+y)} = 4e^{2(x+y)}$ ■

Example 1.2
Suppose $u(x, y) = \sin(x^2 + y^2)$, where $x = te^s$ and $y = s + t$. Find $u_s(s, t)$ and $u_t(s, t)$.

Solution.
We have

$$\begin{aligned} u_s(s,t) =& u_x x_s + u_y y_s = 2x \cos(x^2 + y^2) te^s + 2y \cos(x^2 + y^2) \\ =& [2t^2 e^{2s} + 2(s+t)] \cos[t^2 e^{2s} + (s+t)^2]. \end{aligned}$$

Likewise,

$$\begin{aligned} u_t(s,t) =& u_x x_t + u_y y_t = 2x \cos(x^2 + y^2) e^s + 2y \cos(x^2 + y^2) \\ =& [2te^{2s} + 2(s+t)] \cos[t^2 e^{2s} + (s+t)^2] \; ■ \end{aligned}$$

A **differential equation** is an equation that involves an unknown scalar function (the dependent variable) and one or more of its derivatives. For example,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = -3 \tag{1.1}$$

or

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 0. \tag{1.2}$$

If the unknown function is a function in one single variable then the differential equation is called an **ordinary differential equation**, abbreviated by ODE. An example of an ordinary differential equation is Equation (1.1). In contrast, when the unknown function is a function of two or more independent variables then the differential equation is called a **partial differential equation**, in short PDE. Equation (1.2) is an example of a partial differential equation. In this book we will be focusing on partial differential equations.

Example 1.3
Identify which variables are dependent and which are independent for the following differential equations.
(a) $\frac{d^4y}{dx^4} - x^2 + y = 0.$
(b) $u_{tt} + xu_{tx} = 0.$
(c) $x\frac{dx}{dt} = 4.$
(d) $\frac{\partial y}{\partial u} - 4\frac{\partial y}{\partial v} = u + 3y.$

Solution.
(a) Independent variable is x and the dependent variable is y.
(b) Independent variables are x and t and the dependent variable is u.
(c) Independent variable is t and the dependent variable is x.
(d) Independent variables are u and v and the dependent variable is y ■

Example 1.4
Classify the following as either ODE or PDE.
(a) $u_t = c^2u_{xx}$.
(b) $y'' - 4y' + 5y = 0.$
(c) $z_t + cz_x = 5.$

Solution.
(a) A PDE with dependent variable u and independent variables t and x.
(b) An ODE with dependent variable y and independent variable x.
(c) A PDE with dependent variable z and independent variables t and x ■

The **order** of a differential equation is the highest order derivative occurring in the equation. Thus, (1.1) and (1.2) are second order differential equations.

Example 1.5
Find the order of each of the following partial differential equations:
(a) $xu_x + yu_y = x^2 + y^2$
(b) $uu_x + u_y = 2$
(c) $u_{tt} - c^2u_{xx} = f(x, t)$
(d) $u_t + uu_x + u_{xxx} = 0$
(e) $u_{tt} + u_{xxxx} = 0.$

Solution.
(a) First order (b) First order (c) Second order (d) Third order (e) Fourth

order ■

A first order partial differential equation is called **quasi-linear** if it can be written in the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{1.3}$$

If $a(x, y, u) = \alpha(x, y)$ and $b(x, y, u) = \beta(x, y)$ then (1.3) is called **semi-linear**.
If futhermore, $c(x, y, u) = \gamma(x, y)u + \delta(x, y)$ then (1.3) is called **linear**.
In a similar way, a second order quasi-linear pde has the form
$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d(x, y, u, u_x, u_y)$.
The semi-linear case has the form

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y, u, u_x, u_y).$$

and the linear case is

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y).$$

Note that linear and semi-linear partial differential equations are special cases of quasi-linear equations. However, a quasi-linear PDE needs not be linear: A partial differential equation that is not linear is called **non-linear**. For example, $u_x^2 + 2u_{xy} = 0$ is non-linear. Note that this equation is quasi-linear and semi-linear.
As for ODEs, linear PDEs are usually simpler to analyze/solve than non-linear PDEs.

Example 1.6
Determine whether the given PDE is linear, quasi-linear, semi-linear, or non-linear:
(a) $xu_x + yu_y = x^2 + y^2$.
(b) $uu_x + u_y = 2$.
(c) $u_{tt} - c^2 u_{xx} = f(x, t)$.
(d) $u_t + uu_x + u_{xxx} = 0$.
(e) $u_{tt}^2 + u_{xx} = 0$.

Solution.
(a) Linear, quasi-linear, semi-linear.
(b) Quasi-linear, non-linear.
(c) Linear, quasi-linear, semi-linear.

(d) Quasi-linear, semi-linear, non-linear.
(e) Non-linear ■

A more precise definition of a linear differential equation begins with the concept of a **linear differential operator** L. The operator L is assembled by summing the basic partial derivative operators, with coefficients depending on the independent variables only. The operator acts on sufficiently smooth functions[2] are depending on the relevant independent variables. **Linearity** imposes two key requirements:

$$L[u+v] = L[u] + L[v] \quad \text{and} \quad L[\alpha u] = \alpha L[u],$$

for any two (sufficiently smooth) functions u, v and any constant (a scalar) α.

Example 1.7
Define a linear differential operator for the PDE

$$u_t = c^2 u_{xx}.$$

Solution.
Let $L[u] = u_t - c^2 u_{xx}$. Then one can easily check that $L[u+v] = L[u] + L[v]$ and $L[\alpha u] = \alpha L[u]$ ■

A linear partial differential equation is called **homogeneous** if every term in the equation is a product of a function of the indepedent variables times the dependent function or its derivatives. That is, there are no terms consisting only of the independent variables. A linear partial differential equation that is not homogeneous is called **non-homogeneous.** In this case, there is a term in the equation that involves only the independent variables.
A homogeneous linear partial differential equation has the form

$$L[u] = 0$$

where L is a linear differential operator and the non-homogeneous case has the form

$$L[u] = f(x, y, \cdots).$$

[2]**Smooth functions** are functions that are continuously differentiable up to a certain order.

Example 1.8
Determine whether the equation is homogeneous or non-homogeneous:
(a) $xu_x + yu_y = x^2 + y^2$.
(b) $u_{tt} = c^2 u_{xx}$.
(c) $y^2 u_{xx} + xu_{yy} = 0$.

Solution.
(a) Non-homogeneous because of $x^2 + y^2$.
(b) Homogeneous.
(c) Homogeneous ■

Practice Problems

Problem 1.1
Classify the following equations as either ODE or PDE.

(a) $(y''')^4 + \frac{t^2}{(y')^2+4} = 0.$

(b) $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{y-x}{y+x}.$

(c) $y'' - 4y = 0.$

Problem 1.2
Write the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

in the coordinates $s = x,\ t = x - y.$

Problem 1.3
Write the equation

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0$$

in the coordinates $s = x + y,\ t = 2x.$

Problem 1.4
For each of the following PDEs, state its order and whether it is linear or non-linear. If it is linear, also state whether it is homogeneous or non-homogeneous:
(a) $uu_x + x^2u_{yyy} + \sin x = 0.$
(b) $u_x + e^{x^2}u_y = 0.$
(c) $u_{tt} + (\sin y)u_{yy} - e^t \cos y = 0.$

Problem 1.5
For each of the following PDEs, determine its order and whether it is linear or not. For linear PDEs, state also whether the equation is homogeneous or not. For non-linear PDEs, circle all term(s) that are not linear.
(a) $x^2u_{xx} + e^xu = xu_{xyy}.$
(b) $e^yu_{xxx} + e^xu = -\sin y + 10xu_y.$
(c) $y^2u_{xx} + e^xuu_x = 2xu_y + u.$
(d) $u_xu_{xxy} + e^xuu_y = 5x^2u_x.$
(e) $u_t = k^2(u_{xx} + u_{yy}) + f(x, y, t).$

Problem 1.6
Which of the following PDEs are linear?
(a) **Laplace's equation:** $u_{xx} + u_{yy} = 0.$
(b) **Convection (transport) equation:** $u_t + cu_x = 0.$
(c) **Minimal surface equation:** $(1+Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1+Z_x^2)Z_{yy} = 0.$
(d) **Korteweg-Vries equation:** $u_t + 6uu_x = u_{xxx}.$

Problem 1.7
Classify the following differential equations as ODEs or PDEs, linear or non-linear, and determine their order. For the linear equations, determine whether or not they are homogeneous.
(a) The **diffusion equation** for $u(x,t)$:

$$u_t = ku_{xx}.$$

(b) The **wave equation** for $w(x,t)$:

$$w_{tt} = c^2 w_{xx}.$$

(c) The **thin film equation** for $h(x,t)$:

$$h_t = -(hh_{xxx})_x.$$

(d) The **forced harmonic oscillator** for $y(t)$:

$$y_{tt} + \omega^2 y = F\cos(\omega t).$$

(e) The **Poisson Equation** for the electric potential $\Phi(x,y,z)$:

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 4\pi\rho(x,y,z)$$

where $\rho(x,y,z)$ is a known charge density.
(f) **Burger's equation** for $h(x,t)$:

$$h_t + hh_x = \nu h_{xx}.$$

Problem 1.8
Write down the general form of a linear second order differential equation of a function in three variables.

Problem 1.9
Give the orders of the following PDEs, and classify them as linear or non-linear. If the PDE is linear, specify whether it is homogeneous or non-homogeneous.
(a) $x^2u_{xxy} + y^2u_{yy} - \log(1 + y^2)u = 0$.
(b) $u_x + u^3 = 1$.
(c) $u_{xxyy} + e^x u_x = y$.
(d) $uu_{xx} + u_{yy} - u = 0$.
(e) $u_{xx} + u_t = 3u$.

Problem 1.10
Consider the second-order PDE

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

Use the change of variables $v(x, y) = y - 2x$ and $w(x, y) = x$ to show that $u_{ww} = 0$.

Problem 1.11
Write the one dimensional wave equation $u_{tt} = c^2u_{xx}$ in the coordinates $v = x + ct$ and $w = x - ct$.

Problem 1.12
Write the PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$

in the coordinates $v(x, y) = y - 3x$ and $w(x, y) = x + y$.

Problem 1.13
Write the PDE

$$au_x + bu_y = 0, \ a \neq 0$$

in the coordinates $s(x, y) = bx - ay$ and $t(x, y) = x$.

Problem 1.14
Write the PDE

$$u_x + u_y = 1$$

in the coordinates $s = x - y$ and $t = x$.

Problem 1.15
Write the PDE

$$au_t + bu_x = u, \quad b \neq 0$$

in the coordinates $v = ax - bt$ and $w = x$.