Arkansas Tech University
MATH 3243: Differential Equations I (Fall 2021)
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8.4 Exponential Matrix

In this section we look at a different way for solving the homogeneous system $\mathbf{X}' = \mathbf{AX}$ that involves the concept of exponential matrix which we define next.
For any $n \times n$ matrix $\mathbf{A}$ of constant entries, we define the **exponential matrix**

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^nt^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{\mathbf{A}^nt^n}{n!}$$

where $\mathbf{I}$ is the matrix with 1 on the main diagonal and 0 elsewhere. Also, $\mathbf{A}^n = \mathbf{A}(\mathbf{A}^{n-1})$.

Example 8.4.1
Find $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Solution.
We have

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2^1 \end{bmatrix} \\
\mathbf{A}^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2^2 \end{bmatrix} \\
\mathbf{A}^3 &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2^3 \end{bmatrix} \\
\mathbf{A}^4 &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2^4 \end{bmatrix}
\end{aligned}$$

Inductively, we have

$$\mathbf{A}^k = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2^{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix}.$$

Hence,

$$
\begin{aligned}
e^{\mathbf{A}t} =& \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^nt^n}{n!} + \cdots \\
=& \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \frac{t}{1!} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{t^2}{2!} + \cdots \\
=& \begin{bmatrix} 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots & 0 \\ 0 & 1 + \frac{(2t)}{1!} + \frac{(2t)^2}{2!} + \cdots \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \blacksquare
\end{aligned}
$$

Next, we find the derivative of $e^{\mathbf{A}t}$. We have

$$
\begin{aligned}
\frac{d}{dt}e^{\mathbf{A}t} =& \frac{d}{dt}\left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^nt^n}{n!} + \cdots\right] \\
=& \mathbf{A} + \mathbf{A}^2t + \mathbf{A}^3\frac{t^2}{2!} + \mathbf{A}^4\frac{t^3}{3!} \cdots \\
=& \mathbf{A}\left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots\right] = \mathbf{A}e^{\mathbf{A}t}.
\end{aligned}
$$

Next, we will show that $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution to the homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$. Indeed,

$$\mathbf{X}' = \mathbf{A}e^{\mathbf{A}t}\mathbf{C} = \mathbf{A}\mathbf{X}.$$

Example 8.4.2

Using exponential matrices, find the general solution to $\mathbf{X}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{X}$.

Solution.

We have

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1\begin{bmatrix} e^t \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} \blacksquare$$

Exponential matrices can be used to solve the non-homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$. The complementary solution is found as above and the particular solution to the non-homoegenous system is

$$\mathbf{X}_p = e^{\mathbf{A}t}\int_{t_0}^t e^{-\mathbf{A}s}\mathbf{F}(s)ds.$$

Indeed,

$$\mathbf{X}_p' = e^{\mathbf{A}t}e^{-\mathbf{A}t}\mathbf{F}(t) + \mathbf{A}e^{\mathbf{A}t}\int_{t_0}^t e^{-\mathbf{A}s}\mathbf{F}(s)ds = \mathbf{A}\mathbf{X}_p + \mathbf{F}.$$

Example 8.4.3
Using exponential matrices, find the general solution to

$$\mathbf{X}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Solution.
From Example 8.4.2, we found

$$\mathbf{X}_c = c_1 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \end{bmatrix}.$$

The complementary solution is

$$\begin{aligned}
\mathbf{X}_p &= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} ds \\
&= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \int_0^t \begin{bmatrix} 3e^{-s} \\ -e^{-2s} \end{bmatrix} ds \\
&= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -3e^{-s} \\ \frac{1}{2}e^{-2s} \end{bmatrix} \Bigg|_0^t \\
&= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -3e^{-t} + 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{bmatrix} \Bigg|_0^t \\
&= \begin{bmatrix} -3 + 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{bmatrix}.
\end{aligned}$$

Hence, the general solution is

$$\mathbf{X}(t) = \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \end{bmatrix} + \begin{bmatrix} -3 + 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix} \blacksquare$$

Finding $e^{\mathbf{A}t}$ Using the Laplace Transform
We can find $e^{\mathbf{A}t}$ by using Laplace transform. Indeed, $\mathbf{X} = e^{\mathbf{A}t}$ satisfies the initial value problem $\mathbf{X}' = \mathbf{A}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}$. Moreover,

$$s\mathcal{L}(\mathbf{X}) - \mathbf{X}(0) = A\mathcal{L}(\mathbf{X}) \Rightarrow (s\mathbf{I} - \mathbf{A})\mathcal{L}(\mathbf{X}) = \mathbf{I}.$$

Thus,

$$\mathbf{X} = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}].$$

Example 8.4.4

Use the Laplace transform to compute $e^{\mathbf{A}t}$ where $\left[\begin{array}{cc} 4 & 3 \\ -4 & -4 \end{array}\right]$.

Solution.

We have

$$\begin{aligned}
s\mathbf{I}-\mathbf{A} &= \left[\begin{array}{cc} s-4 & -3 \\ 4 & s+4 \end{array}\right] \\
(s\mathbf{I}-\mathbf{A})^{-1} &= \left[\begin{array}{cc} \frac{s+4}{s^2-4} & \frac{3}{s^2-4} \\ -\frac{4}{s^2-4} & \frac{s-4}{s^2-4} \end{array}\right] \\
&= \left[\begin{array}{cc} \frac{\frac{3}{2}}{s-2} - \frac{\frac{1}{2}}{s+2} & \frac{\frac{3}{4}}{s-2} - \frac{\frac{3}{4}}{s+2} \\ -\frac{1}{s-2} + \frac{1}{s+2} & \frac{-\frac{1}{2}}{s-2} + \frac{\frac{3}{2}}{s+2} \end{array}\right] \\
e^{\mathbf{A}t} &= \mathcal{L}^{-1}\left(\left[\begin{array}{cc} \frac{\frac{3}{2}}{s-2} - \frac{\frac{1}{2}}{s+2} & \frac{\frac{3}{4}}{s-2} - \frac{\frac{3}{4}}{s+2} \\ -\frac{1}{s-2} + \frac{1}{s+2} & \frac{-\frac{1}{2}}{s-2} + \frac{\frac{3}{2}}{s+2} \end{array}\right]\right) \\
&= \left[\begin{array}{cc} \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} & \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t} \\ -e^{2t} + e^{-2t} & -\frac{1}{2}e^{2t} + \frac{3}{2}e^{-2t} \end{array}\right] \blacksquare
\end{aligned}$$