Arkansas Tech University
MATH 3243: Differential Equations I (Fall 2021)
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8.3 Non Homogeneous Linear Systems

As in the case of a single non-homogeneous linear equation, the general solution to the non-homogenous linear system

$$X' = AX + F \tag{8.3.1}$$

is the sum of the general solution of the homogeneous system and a particular solution of the non-homogeneous system. That is,

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

where $\mathbf{X}_c$ is the **complementary solution** and $\mathbf{X}_p$ is the particular solution. One way for finding $\mathbf{X}_p$ is by means of the so-called the **method of variation of parameters** which we discuss next.
Let the general solution to the homogeneous system be

$$\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n = c_1\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} + c_2\begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

which can be written as

$$\mathbf{X}_c = \begin{bmatrix} c_1x_{11} + c_2x_{12} + \cdots + c_nx_{1n} \\ c_1x_{21} + c_2x_{22} + \cdots + c_nx_{2n} \\ \vdots \\ c_1x_{n1} + c_2x_{n2} + \cdots + c_nx_{nn} \end{bmatrix} = \mathbf{\Phi}(t)\mathbf{C}$$

where

$$\mathbf{\Phi}(t) = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We call $\mathbf{\Phi}(t)$ the **fundamental matrix** of the homogeneous system. The following observations about this matrix are important. First, $\mathbf{\Phi}(t)$ is the Wronskian of $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$. Since these vectors are linearly independent, for the system $\mathbf{\Phi}(t)\mathbf{X} = \mathbf{0}$ to have a trivial solution the matrix $\mathbf{\Phi}(t)$ must be invertible[1]. That is, the determinant of $\mathbf{\Phi}(t)$ is non-zero.
Second, the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ implies that $\mathbf{X}'_i = \mathbf{A}\mathbf{X}_i$ for $i = 1, 2, \cdots, n$. Hence,

$$\mathbf{\Phi}'(t) = [\mathbf{X}'_1 \quad \mathbf{X}'_2 \quad \cdots \quad \mathbf{X}'_n] = [\mathbf{A}\mathbf{X}_1 \quad \mathbf{A}\mathbf{X}_2 \quad \cdots \quad \mathbf{A}\mathbf{X}_n] = \mathbf{A}\mathbf{\Phi}(t).$$

The variation of parameters method seeks a solution of the form $\mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t)$ where

$$\mathbf{U}(t) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

is to be found. Since $\mathbf{X}_p$ is a solution to (8.3.1), we have

$$\mathbf{\Phi}(t)\mathbf{U}'(t) + \mathbf{\Phi}'(t)\mathbf{U}(t) = \mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t) + \mathbf{F}(t)$$

or

$$\mathbf{\Phi}(t)\mathbf{U}'(t) + \mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t) = \mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t) + \mathbf{F}(t).$$

Thus,

$$\mathbf{\Phi}(t)\mathbf{U}'(t) = \mathbf{F}(t).$$

Since $\mathbf{\Phi}(t)$ is invertible, the last equation implies

$$\mathbf{U}'(t) = \mathbf{\Phi}^{-1}(t)\mathbf{F}(t) \Rightarrow \mathbf{U}(t) = \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt \Rightarrow \mathbf{X}_p = \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt.$$

Example 8.3.1
Show $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

Solution.
Indeed, we have

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \blacksquare$$

[1] An $n \times n$ matrix $\mathbf{A}$ is **invertible** if and only if $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$, where $\mathbf{I}_n$ is the **identity matrix**.

Example 8.3.2
Solve: $\mathbf{X}' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$.

Solution.
First, we find the complementary solution. The characteristic equation is

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+2)(\lambda+5) = 0.$$

Thus, $\lambda_1 = -2$ and $\lambda_2 = -5$. The eigenvectors of λ_1 satisfy the equation

$$\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which yields the system

$$\begin{aligned} -3x_1 + x_2 &= -\,2x_1 \\ 2x_1 - 4x_2 &= -\,2x_2 \end{aligned}$$

which gives $x_1 = x_2$. Thus, an eigenvector of $\lambda_1 = -2$ is

$$\mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with corresponding solution

$$\mathbf{X}_1 = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}.$$

Likewise, a solution corresponding to $\lambda_2 = -5$ is

$$\mathbf{X}_2 = \begin{bmatrix} e^{-5t} \\ -2e^{-5t} \end{bmatrix}.$$

Hence, using the previous example, we find

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \Rightarrow \mathbf{\Phi}^{-1}(t) \begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\mathbf{X}_p =&\mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \int \begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix} \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix} dt \\
=& \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \int \begin{bmatrix} 2te^{2t} + \frac{1}{3}e^{t} \\ te^{5t} - \frac{1}{3}e^{4t} \end{bmatrix} dt \\
=& \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \begin{bmatrix} te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t} \\ \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \end{bmatrix} \\
=& \begin{bmatrix} \frac{6}{5}t + \frac{1}{4}e^{-t} - \frac{27}{50} \\ \frac{3}{5}t + \frac{1}{2}e^{-t} - \frac{21}{50} \end{bmatrix}.
\end{aligned}$$

The general solution to the non-homogeneous system is

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t} + \begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \end{bmatrix} t - \begin{bmatrix} \frac{27}{50} \\ \frac{21}{50} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} e^{-t} \blacksquare$$

Remark 8.3.1
If an initial value is given, say $\mathbf{X}(t_0) = \mathbf{X}_0$ then the indefinite integral $\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt$ is being replaced by $\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)ds$.