Arkansas Tech University
MATH 3243: Differential Equations I (Fall 2021)
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8.2 Homogeneous Linear Systems

Given an $n \times n$ matrix A and a constant λ. The equation

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

is called the **characteristic equation** of the matrix A; its solutions are called the **eigenvalues of A**.
If λ is an eigenvalue of A and $\mathbf{x}$ is a $n \times 1$ non-zero vector such that $A\mathbf{x} = \lambda\mathbf{x}$ then we call $\mathbf{x}$ and **eigenvector** of λ.
Now, consider a homogeneous system $X' = AX$. We seek solutions of the form $X = e^{\lambda t}\mathbf{K}$ where $\mathbf{K}$ is a non-zero vector. Thus, $\lambda e^{\lambda t}\mathbf{K} = e^{\lambda t}A\mathbf{K}$ which can be written as

$$(A - \lambda I)\mathbf{K} = 0 \qquad (8.2.1)$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

For the system in (8.2.1) to have a non-trivial solution, λ must be an eigenvalue of A.

Distinct Real Eigenvalues

If the matrix A has n distinct real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ with corresponding eigenvectors $\mathbf{K_1}, \mathbf{K_2}, \cdots, \mathbf{K_n}$ then the vectors

$$\mathbf{X_1} = e^{\lambda_1 t}\mathbf{K_1},\ \mathbf{X_2} = e^{\lambda_2 t}\mathbf{K_2},\ \cdots,\ \mathbf{X_n} = e^{\lambda_n t}\mathbf{K_n}$$

form a fundamental set of solutions to the homogeneous system and the general solution is given by

$$\mathbf{X}(t) = c_1 e^{\lambda_1 t}\mathbf{K_1} + c_2 e^{\lambda_2 t}\mathbf{K_2} + \cdots + c_n e^{\lambda_n t}\mathbf{K_n}.$$

Example 8.2.1
Solve the initial value problem

$$\mathbf{X}' = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}\mathbf{X}, \quad \mathbf{X}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Solution.
We first find the eigenvalues of A by solving the characteristic equation

$$\begin{vmatrix} \frac{1}{2} - \lambda & 0 \\ 1 & -\frac{1}{2} - \lambda \end{vmatrix} = -(\frac{1}{2} - \lambda)(\frac{1}{2} + \lambda) = 0$$

and so the eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$.
Now for $\lambda_1 = \frac{1}{2}$ the equation

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

gives $k_1 - k_2 = 0$. Choose $k_1 = 1$ so that $k_2 = 1$ and

$$\mathbf{K_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Likewise, for $\lambda_2 = -\frac{1}{2}$ the equation

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

gives $k_1 = 0$ and k_2 arbitrary. Choose $k_2 = 1$ so that

$$\mathbf{K_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, the general solution is

$$\mathbf{X}(t) = c_1 e^{\frac{t}{2}}\mathbf{K_1} + c_2 e^{-\frac{t}{2}}\mathbf{K_2}.$$

Using the initial condition, we find

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 + c_2 \end{bmatrix}$$

which implies $c_1 = 3$ and $c_1 + c_2 = 5$. That is, $c_1 = 3$ and $c_2 = 2$. The solution to the initial value problem is

$$\mathbf{X}(t) = 3e^{\frac{t}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-\frac{t}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \blacksquare$$

Repeated Eigenvalues

When the characteristic equation has a factor of the form $(a - \lambda)^n$ then we say that λ is an eigen value of multiplicity n. In the case $n = 2$, one solution is $\mathbf{X_1} = e^{\lambda t}\mathbf{K}$ and a second solution is $\mathbf{X_2} = e^{\lambda t} t\mathbf{K} + e^{\lambda t}\mathbf{P}$. Note that since this is a solution to the homogeneous equation, by substitution we find

$$(A\mathbf{K} - \lambda \mathbf{K})te^{\lambda t} + (A\mathbf{P} - \lambda \mathbf{P} - \mathbf{K})e^{\lambda t} = \mathbf{0}$$

which implies that $\mathbf{P}$ satisfies the equation $(A - \lambda \mathbf{I})\mathbf{P} = \mathbf{K}$.

Example 8.2.2

Solve

$$\mathbf{X}' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \mathbf{X}.$$

Solution.

The characteristic equation is

$$\begin{vmatrix} 3 - \lambda & -18 \\ 2 & -9 - \lambda \end{vmatrix} = (\lambda + 3)^2 = 0$$

so that $\lambda = -3$ is of multiplicity 2. For $\lambda = -3$, we have

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $6k_1 - 18k_2 = 0$ and $2k_1 - 6k_2 = 0$. Solving, we find $k_2 = 3k_2$. Letting $k_2 = 1$ we find $k_1 = 3$. Hence,

$$\mathbf{K} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

To find $\mathbf{P}$ we solve the system

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

which yields $6p_1 - 18p_2 = 3$ and $2p_1 - 6p_2 = 1$. Solving this system, we find $6p_1 - 18p_2 = 3$. Letting $p_2 = 0$ we find $p_1 = \frac{1}{2}$. Hence,

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

and the general solution is

$$\mathbf{X} = c_1 e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \left(te^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \right) \blacksquare$$

Complex Eigenvalues

By a complex number we mean a number that can be written in the form $\lambda = \alpha + i\beta$ where $i = \sqrt{-1}$. We call α the **real part** pf λ and β the **imaginary part**. Also, the **conjugate** of $\lambda = \alpha + i\beta$ is the complex number $\overline{\lambda} = \alpha - i\beta$. We recall that complex solutions to an algebraic equation always appear in conjugate pairs. That is, if $\lambda_1 = \alpha + i\beta$ is a solution to an equation then $\lambda_2 = \alpha - i\beta$ is also a solution.

Now, if $\lambda = \alpha + i\beta$ is an eigenvalue of A with complex eigenvector $\mathbf{K}$ then the general solution to the homogeneous equation is given

$$X(t) = c_1 e^{\alpha t}[\mathbf{B_1} \cos \beta t - \mathbf{B_2} \sin \beta t] + c_2 e^{\alpha t}[\mathbf{B_2} \cos \beta t + \mathbf{B_1} \sin \beta t]$$

where $\mathbf{B_1}$ is the real part of $\mathbf{K}$ and $\mathbf{B_2}$ is the imaginary part of $\mathbf{K}$.

Example 8.2.3

Solve

$$\mathbf{X}' = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix} \mathbf{X}.$$

Solution.

The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

so that $\lambda_1 = 2i$ and $\lambda_2 = -2i$. For $\lambda_1 = 2i$, we have

$$\begin{bmatrix} 2-2i & 8 \\ -1 & -2-2i \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $k_1 = -2(1+i)k_2$. Letting $k_2 = -1$ we find $k_1 = 2+2i$. Hence,

$$\mathbf{K} = \begin{bmatrix} 2+2i \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{B_1} = \mathrm{Re}(\mathbf{K}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } \mathbf{B_2} = \mathrm{Im}(\mathbf{K}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The general solution to the given system is

$$\begin{aligned} \mathbf{X} =& c_2 \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t \right) + c_2 \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \sin 2t \right) \\ =& c_1 \begin{bmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{bmatrix} \blacksquare \end{aligned}$$