Arkansas Tech University
MATH 3243: Differential Equations I (Fall 2021)
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8.1 First Order Linear Systems

A **linear system of first order differential equations** is a system of the form

$$\begin{aligned}\frac{dx_1}{dt} =& a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\ \frac{dx_2}{dt} =& a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\ &\vdots \\ \frac{dx_n}{dt} =& a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_1(t)\end{aligned} \qquad (8.1.1)$$

where the functions $a_{ij}(t)$ and $f_i(t)$ are continuous on a common interval. If $f_i(t) = 0$ for all t and $i = 1, 2, \cdots, n$ then the linear system (8.1.1) is said to be **homogeneous**; otherwise, it is non-homogeneous.
The system (8.1.1) can be written in **matrix form**

$$X' = AX + F \qquad (8.1.2)$$

where

$$A)(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \ F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

In the case of a homogeneous system, we have

$$X' = AX. \qquad (8.1.3)$$

Example 8.1.1
Write the system

$$\begin{aligned}\frac{dx}{dt} =& 6x + y + z + t\\ \frac{dy}{dt} =& 8x + 7y - z + 10t\\ \frac{dz}{dt} =& 2x + 9y - z + 6t\end{aligned}$$

in matrix form.

Solution.
The matrix form of the non-homogenous system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 6 & 1 & 1 \\ 8 & 7 & -1 \\ 2 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ 10t \\ 6t \end{bmatrix} \blacksquare$$

A **solution vector** on an interval I is a column matrix

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

satisfying the system (8.1.2) on the interval I.

Example 8.1.2
Show that the column matrices

$$X_1(t) = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}, \ X_2(t) = \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}$$

are solution vectors to the system

$$X' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} X.$$

Solution.
We have

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} X_1 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} - 3e^{-2t} \\ 5e^{-2t} - 3e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{bmatrix} X_1'$$

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} X_2 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix} = \begin{bmatrix} 3e^{6t} + 15e^{6t} \\ 15e^{6t} + 15e^{6t} \end{bmatrix} = \begin{bmatrix} 18e^{6t} \\ 30e^{6t} \end{bmatrix} X_2' \blacksquare$$

Now, if t_0 belongs to I then we define the **initial value problem**

$$X' = AX + F, \quad X(t_0) = X_0 \tag{8.1.4}$$

where

$$X_0 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

That is, $x_i(t_0) = c_i$ for $i = 1, 2, \cdots, n$.
Like the case of a linear differential equation, we have the following **superposition principle**:
If $X_1, X_2, \cdots$, are solutions to (8.1.3) then for any scalars $c_1, c_2, \cdots, c_n$, the column vector

$$X = c_1X_1 + c_2X_2 + c_nX_n$$

which is a **linear combination** of $X_1, X_2, \cdots, X_n$ is a solution to (8.1.3).
As in the discussion of the theory of a single ordinary differentail equation, we introduce the concepts of linear dependence and Wronskian. We say that the solution vectors $X_1, X_2, \cdots, X_n$ tp (8.1.3) are **linearly dependent** on the interval I if there exist constants $c_1, c_2, \cdots, c_n$ not all zero such that

$$c_1X_1 + c_2X_2 + \cdots + c_nX_n = \mathbf{0}$$

for all t in I, where $\mathbf{0}$ is the zero column vector. That is, if one vector is a linear combination of the remaining vectors. If the set of vectors are not linearly depndent on the interval I, it is said to be **linearly independent**. The solution vectors to (8.1.3) can be trested for independence via the **Wronskian** formula

$$W(X_1, X_2, \cdots, X_n) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$

where

$$X_i = \begin{bmatrix} x_{1i}(t) \\ x_{2i}(t) \\ \vdots \\ x_{ni}(t) \end{bmatrix}, \quad i = 1, 2, \cdots, n.$$

Indeed, if t_0 is a number in I and $W(X_1(t_0), X_2(t_0), \cdots, X_n(t_0)) \neq 0$ then the solution vectors $X_1, X_2, \cdots, X_n$ are linearly independent. In this case, we call the set $\{X_1, X_2, \cdots, X_n\}$ a **fundemental set** of the homogeneous system (8.1.3) on the interval I. The **general solution** to (8.1.3) is

$$X(t) = c_1X_1(t) + c_2X_2(t) + \cdots + c_nX_n(t).$$

Example 8.1.3
(a) Show that the matrices

$$X_1(t) = \begin{bmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{bmatrix}, \; X_2(t) = \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}, \; X_3(t) = \begin{bmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ \cos t - \sin t \end{bmatrix}$$

are solutions to the system

$$X' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} X.$$

(b) Show that X_1, X_2, X_3 are lienearly indepdent solutions.
(c) Find the general solution to the given system.

Solution.
(a) We have

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} X_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{bmatrix} = \begin{bmatrix} \cos t - \cos t - \sin t \\ \cos t - \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -2\cos t + \cos t + \sin t \end{bmatrix}$$

$$= \begin{bmatrix} -\sin t \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t + \sin t \end{bmatrix} = X_1'$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} X_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix} = X_2'$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} X_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ \cos t - \sin t \end{bmatrix} = \begin{bmatrix} \sin t - \sin t + \cos t \\ \sin t - \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -2\sin t - \cos t + \sin t \end{bmatrix}$$

$$= \begin{bmatrix} \cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t - \cos t \end{bmatrix} = X_3'.$$

(b) We have

$$W(X_1, X_2, X_3) = \begin{vmatrix} \cos t & 0 & \sin t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t & e^t & -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t & 0 & -\sin t + \cos t \end{vmatrix} = e^t \neq 0.$$

Hence, X_1, X_2, X_3 are linearly independent.
(c) From (a) and (b), we conclude that X_1, X_2, and X_3 form a fundamental set of solutions on $(-\infty, \infty)$. The general solution to the given system is

$$X(t) = c_1 \begin{bmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \sin t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ \cos t - \sin t \end{bmatrix} \blacksquare$$

The **general solution** to the non-homogeneous system (8.1.2) is

$$X(t) = X_c(t) + X_p(t)$$

where $X_c(t)$ is the general solution to (8.1.3) and X_p is a particular solution to (8.1.2).

Example 8.1.4
(a) Show that the column matrices in Example 8.1.2, form a fundamental set to the given system on $(-\infty, \infty)$.
(b) Show that the column matrix $X_p = \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix}$ is a particular solution to the nonhomogeneous syste

$$X' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} X + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix}.$$

(c) Find the general solution to the given system.

Solution.
(a) We have

$$W(X_1, X_2) = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} + 5e^{6t} & \end{vmatrix} = 8e^{4t} \neq 0.$$

Thus, X_1 and X_2 are linearly independent and form a fundamental set.
(b) We have

$$\begin{aligned}
\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} X_p + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix} &= \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3t - 4 \\ -5t + 6 \end{bmatrix} + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} 3t - 4 - 15t + 18 \\ 15t - 20 - 15t + 18 \end{bmatrix} + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -5 \end{bmatrix} = X_p'.
\end{aligned}$$

(c) The general solution to the non-homogeneous system is

$$X(t) = c_1 \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{6t} \\ 5^{6t} \end{bmatrix} + \begin{bmatrix} 12t - 11 \\ -3 \end{bmatrix} \blacksquare$$