Arkansas Tech University
MATH 3243: Differential Equations I (Fall 2021)
Dr. Marcel B Finan

7.5 The Laplace Transform of the Dirac Delta Function

Mechanical systems are often acted on by an external force. Heaviside functions can be thought of as switches changing a forcing function at specified time. For example, $f(t)h(t-a)$ swiches off to 0 for $t < a$ and switches on to $f(t)$ for $t \geq a$. However, Heaviside functions are really not suited to forcing functions that exert a large force over a small time frame. An example of such a force is the striking of an object by a hammer. In this section, we introduce a function that deals with such kind of forcing functions. Moreover, we will see that the Laplace transform of such a function is 1.

The Dirac Delta Function

For $a > 0$ and $t_0 > 0$, we define the piecewise continuous function

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \leq t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \leq t < t_0 + a \\ 0, & t \geq t_0 + a. \end{cases}$$

The graph of such a function is shown in Figure 7.5.1.

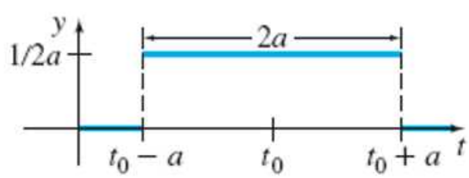

Figure 7.5.1

The area under the graph is 1. Indeed, we have

$$\int_0^\infty \delta_a(t-t_0)dt = \int_{t_0-a}^{t_0+a} \frac{1}{2a}dt = \left.\frac{t}{2a}\right|_{t_0-a}^{t_0+a} = 1.$$

Also, we can express δ_a in terms of the Heaviside function. Indeed,

$$\delta_a(t-t_0) = \frac{1}{2a}[h(t-(t_0-a)) - h(t-(t_0+a))].$$

The behavior of $\delta_a(t - t_0$ as $a \to \infty$ is shown in Figure 7.5.2.

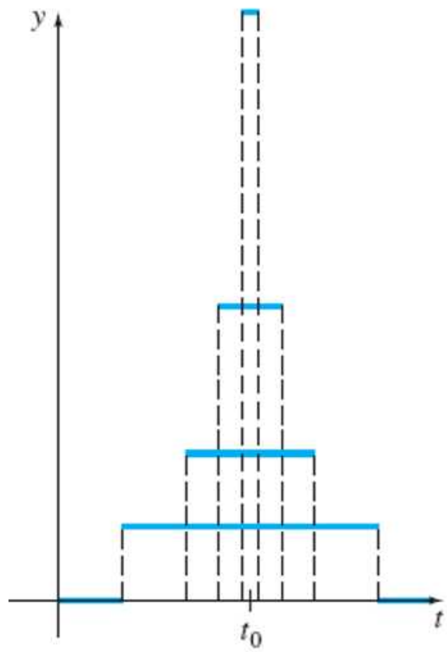

Figure 7.5.2

The **Dirac delta function** is defined as

$$\delta(t - t_0) = \lim_{a \to 0} \delta_a(t - t_0)$$

or explicitly by

$$\delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0. \end{cases}$$

Moreover,

$$\int_0^\infty \delta(t - t_0) = \lim_{a \to 0} \int_0^\infty \delta_a(t - t_0) dt = 1.$$

The Dirac Delta function is not a real function in the conventional sense as no function defined on the real numbers has these properties. It is instead an example of something called a **generalized function** or **distribution.**

Next, we look for the Laplace transform of $\delta(t-t_0)$. We have

$$\begin{aligned}
\mathcal{L}[\delta_a(t-t_0)] =& \frac{1}{2a}[\mathcal{L}[h(t-(t_0-a))] - \mathcal{L}[h(t-(t_0+a))]\\
=& \frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s} - \frac{e^{-s(t_0+a)}}{s}\right] = e^{-st_0}\left[\frac{e^{sa}-e^{-sa}}{2sa}\right]\\
=& e^{-st_0}\frac{\sinh{(sa)}}{as}.
\end{aligned}$$

Thus,

$$\mathcal{L}[\delta(t-t_0)] = \lim_{a\to 0} e^{-st_0}\frac{\sinh{(sa)}}{as} = \lim_{a\to 0} e^{-st_0}\frac{s\cosh{(sa)}}{s} = e^{-st_0}.$$

In particular,

$$\mathcal{L}[\delta(t)] = 1.$$

This last result emphasezes the fact that $\delta(t)$ is not a function since we know that for a function $f(t)$ we must have $\lim_{s\to 0}\mathcal{L}[f(t)] = 0$.

Example 7.5.1
Solve the IVB: $y'' + y = 4\delta(t-2\pi), \quad y(0) = 1, \; y'(0) = 0.$

Solution.
Taking the Laplace transoform of both sides, we find

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}.$$

Using the initial conditions, we find

$$Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}.$$

Taking the inverse Laplace transform, we find

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + 4\mathcal{L}^{-1}\left[\frac{e^{-2\pi s}}{s^2+1}\right] = \cos t + 4\sin{(t-2\pi)}h(t-2\pi) \blacksquare$$