Arkansas Tech University
MATH 3243: Differential Equations I
Dr. Marcel B Finan

7.4 Further Properties of Laplace Transform

In this section we develop several more operational properties of Laplace Transform.

The Derivative of a Transform

If $F(s) = \mathcal{L}[f(t)]$ and $n = 1, 2, 3, \cdots$ then

$$\mathcal{L}[t^f(t)] = (-1)^n \frac{d^n}{ds^n} F(s). \tag{7.4.1}$$

We prove Equation (7.4.1) for $n = 1$ and $n = 2$. We have

$$\begin{aligned}
\frac{d}{ds}F(s) =& \frac{\partial}{\partial s}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}tf(t)]dt \\
=& -\int_0^\infty e^{-st}tf(t)dt = -\mathcal{L}[tf(t)]. \\
\mathcal{L}[t^2 f(t)] =& \mathcal{L}[t \cdot tf(t)] = -\frac{d}{ds}[\mathcal{L}[tf(t)]] \\
=& -\frac{d}{ds}\left[-\frac{d}{ds}F(s)\right] = \frac{d^2}{ds^2}F(s).
\end{aligned}$$

Example 7.4.1

Solve the IVP: $x'' + 16x = \cos(4t), \quad x(0) = 0, \ x'(0) = 1.$

Solution.

Taking the Laplace transform of both sides of the ODE and using the appropriate formulas coupled with the initial conditions, we find

$$(s^2 + 16)X(s) = 1 + \frac{s}{s^2 + 16} \Longrightarrow X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}.$$

Hence,

$$\begin{aligned}
x(t) =& \mathcal{L}^{-1}\left[\frac{1}{s^2 + 16}\right] + \mathcal{L}^{-1}\left[\frac{s}{(s^2 + 16)^2}\right] \\
=& \frac{1}{4}\mathcal{L}^{-1}\left[\frac{4}{s^2 + 4^2}\right] + \frac{1}{8}\mathcal{L}^{-1}\left[\frac{8s}{(s^2 + 16)^2}\right] \\
=& \frac{1}{4}\sin(4t) + \frac{1}{8}t\sin(4t), \ t \geq 0 \ \blacksquare
\end{aligned}$$

The Inverse Laplace of a product: Convolution

Convolution integrals are useful when finding the inverse Laplace transform of products $H(s) = F(s)G(s)$. They are defined as follows: The **convolution** of two scalar piecewise continuous functions $f(t)$ and $g(t)$ defined for $t \geq 0$ is the integral

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

Example 7.4.2

Find $f * g$ where $f(t) = e^{-t}$ and $g(t) = \sin t$.

Solution.

Using integration by parts twice we arrive at

$$\begin{aligned}(f * g)(t) =& \int_0^t e^{-(t-s)} \sin s ds \\ =& \frac{1}{2}\left[e^{-(t-s)}(\sin s - \cos s)\right]_0^t \\ =& \frac{e^{-t}}{2} + \frac{1}{2}(\sin t - \cos t) \blacksquare\end{aligned}$$

Next, we state several properties of convolution product, which resemble those of ordinary product.

Theorem 7.4.1

Let $f(t), g(t)$, and $k(t)$ be three piecewise continuous scalar functions defined for $t \geq 0$ and c_1 and c_2 are arbitrary constants. Then

(i) $f * g = g * f$ (Commutative Law)

(ii) $(f * g) * k = f * (g * k)$ (Associative Law)

(iii) $f * (c_1 g + c_2 k) = c_1 f * g + c_2 f * k$ (Distributive Law)

Proof.

(i) Letting $w = t - s$, We have

$$(f*g)(t) = \int_0^t f(t-s)g(s)ds = -\int_t^0 f(w)g(t-w)dw = \int_0^t g(t-w)f(w)dw = (g*f)(t).$$

(ii) Using (i) and Figure 7.4.1, we have

$$
\begin{aligned}
[(f*g)*k](t) &= \int_0^t (f*g)(t)g(t-s)ds = \int_{s=0}^{s=t}\left[\int_{u=0}^{u=s} f(u)g(s-u)du\right]k(t-s)ds \\
&= \int_{s=0}^{s=t}\int_{u=0}^{u=s} f(u)g(s-u)k(t-s)duds \\
&= \int_{u=0}^{u=t}\int_{s=u}^{s=t} f(u)g(s-u)k(t-s)dsdu \\
&= \int_{u=0}^{u=t} f(u)\left[\int_{s=u}^{s=t} g(s-u)k(t-s)ds\right]du \\
&= \int_{u=0}^{u=t} f(u)\left[\int_{w=0}^{w=t-u} g(w)k(t-u-w)ds\right]du \\
&= \int_{u=0}^{u=t} f(u)[g*k](t-u)du = [f*(g*k)](t).
\end{aligned}
$$

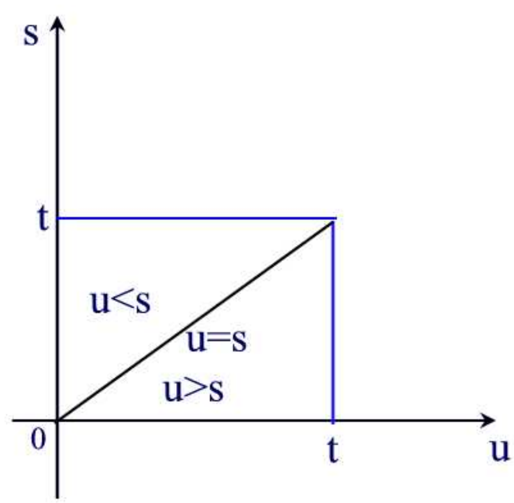

Figure 7.4.1

(iii) We have

$$\begin{aligned}[f*(c_1g+c_2k)](t)=&\int_0^t f(t-s)[(c_1g(s)+c_2k(s))ds\\ =&\int_0^t f(t-s)c_1g(s)ds+\int_0^t f(t-s)c_2k(s)ds\\ =&c_1\int_0^t f(t-s)g(s)ds+c_2\int_0^t f(t-s)k(s)ds\\ =&c_1(f*g)(t)+c_2(f*k)(t) \blacksquare\end{aligned}$$

Example 7.4.3
Express the solution to the initial value problem $y'+\alpha y=g(t),\ y(0)=y_0$ in terms of a convolution integral.

Solution.
Solving this initial value problem by the method of integrating factor we find

$$y(t)=e^{-\alpha t}y_0+\int_0^t e^{-\alpha(t-s)}g(s)ds=e^{-\alpha t}y_0+e^{-\alpha t}*g(t) \blacksquare$$

The following theorem, known as the **Convolution Theorem**, provides a way for finding the Laplace transform of a convolution integral and also finding the inverse Laplace transform of a product.

Theorem 7.4.2
If $f(t)$ and $g(t)$ are piecewise continuous for $t\geq 0$, and of exponential order then

$$\mathcal{L}[(f*g)(t)]=\mathcal{L}[f(t)]\mathcal{L}[g(t)]=F(s)G(s).$$

Thus, $(f*g)(t)=\mathcal{L}^{-1}[F(s)G(s)]$.

Proof.
We have

$$\begin{aligned}F(s)G(s)=&\left(\int_0^\infty e^{-su}f(u)du\right)\left(\int_0^\infty e^{-sw}g(w)dw\right)\\ =&\int_0^\infty\int_0^\infty e^{-s(u+w)}f(u)g(w)dudw\\ =&\int_0^\infty\int_0^\infty e^{-s(u+w)}f(u)g(w)dwdu\end{aligned}$$

$$= \int_0^\infty f(u) \left[\int_u^\infty e^{-st} g(t-u) dt \right] du$$
$$= \int_0^\infty e^{-st} \left[\int_0^t f(u) g(t-u) du \right] dt$$
$$= \int_0^\infty e^{-st} (f * g)(t) dt = \mathcal{L}[(f*g)(t)] \blacksquare$$

Example 7.4.4
Evaluate: $\mathcal{L}[\int_0^t e^u \sin(t-u) du$.

Solution.
We have

$$\mathcal{L}[\int_0^t e^u \sin(t-u) du = \mathcal{L}[e^t * \sin t] = \mathcal{L}[e^t]\mathcal{L}[\sin t]$$
$$= \frac{1}{s-1} \cdot \frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)} \blacksquare$$

Example 7.4.5
Use the convolution theorem to find the inverse Laplace transform of

$$H(s) = \frac{1}{(s^2+a^2)^2}.$$

Solution.
Note that

$$H(s) = \left(\frac{1}{s^2+a^2} \right) \left(\frac{1}{s^2+a^2} \right).$$

So, in this case we have, $F(s) = G(s) = \frac{1}{s^2+a^2}$ so that $f(t) = g(t) = \frac{1}{a} \sin(at)$. Thus,

$$(f * g)(t) = \frac{1}{a^2} \int_0^t \sin(at - as) \sin(as) ds = \frac{1}{2a^3} (\sin(at) - at \cos(at))$$

where we used the trigonometric identity

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)] \blacksquare$$

Example 7.4.6
(a) Show that $\mathcal{L}\left[\int_0^t f(s)ds\right] = \frac{F(s)}{s}$.
(b) Evaluate $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right]$ and $\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$.

Solution.
(a) Let $g(t) = 1$ for all $t \geq 0$. Then by the Convolution Theorem, we have

$$\mathcal{L}\left[\int_0^t f(s)ds\right] = \mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = \frac{F(s)}{s}.$$

(b) We have

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] =& \mathcal{L}^{-1}\left[\frac{\frac{1}{s^2+1}}{s}\right] \\
=& \int_0^t \sin s ds = 1 - \cos t \\
\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] =& \mathcal{L}^{-1}\left[\frac{\frac{1}{s(s^2+1)}}{s}\right] \\
=& \int_0^t (1 - \cos s)ds = t - \sin t \blacksquare
\end{aligned}$$

Example 7.4.7
Solve the initial value problem

$$4y'' + y = g(t), \quad y(0) = 3, \; y'(0) = -7$$

Solution.
Take the Laplace transform of all the terms and plug in the initial conditions to obtain

$$4(s^2Y(s) - 3s + 7) + Y(s) = G(s)$$

or

$$(4s^2 + 1)Y(s) - 12s + 28 = G(s).$$

Solving for $Y(s)$ we find

$$\begin{aligned}
Y(s) =& \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)} \\
=& \frac{3s}{s^2 + \left(\frac{1}{2}\right)^2} - 14\frac{\frac{1}{2}}{s^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{2}G(s)\frac{\frac{1}{2}}{s^2 + \left(\frac{1}{2}\right)^2}
\end{aligned}$$

Hence,

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 14\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{s}{2}\right)g(t-s)ds.$$

So, once we decide on a $g(t)$ all we need to do is to evaluate the integral and we'll have the solution ■

Example 7.4.8
Use Laplace transform to solve for $y(t)$:

$$y(t) - \int_0^t e^{(t-\lambda)}y(\lambda)d\lambda = t.$$

Solution.
Note that the given equation reduces to $e^t * y(t) = y(t) - t$. Taking Laplace transform of both sides we find $\frac{Y(s)}{s-1} = Y(s) - \frac{1}{s^2}$. Solving for $Y(s)$ we find $Y(s) = \frac{s-1}{s^2(s-2)}$. Using partial fractions decomposition we can write

$$\frac{s-1}{s^2(s-2)} = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{(s-2)}.$$

Hence,

$$y(t) = -\frac{1}{4} + \frac{t}{2} + \frac{1}{4}e^{2t}, t \geq 0 \blacksquare$$

Example 7.4.9
Solve the following initial value problem.

$$y' - y = \int_0^t (t-\lambda)e^{\lambda}d\lambda, \quad y(0) = -1.$$

Solution.
Note that $y' - y = t * e^t$. Taking Lalplace transform of both sides we find $sY - (-1) - Y = \frac{1}{s^2} \cdot \frac{1}{s-1}$. This implies that $Y(s) = -\frac{1}{s-1} + \frac{1}{s^2(s-1)^2}$. Using partial fractions decomposition we can write

$$\frac{1}{s^2(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}.$$

Thus,

$$Y(s) = -\frac{1}{s-1} + \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{1}{(s-1)^2}.$$

Finally,

$$y(t) = 2 + t - 3e^t + te^t, t \geq 0 \blacksquare$$

Transform of a Periodic Function

In many applications, the nonhomogeneous term in a linear differential equation is a periodic function. In this section, we derive a formula for the Laplace transform of such periodic functions.

Recall that a function $f(t)$ is said to be $T-$**periodic** if T is the smallest positive interger such that $f(t+T) = f(t)$ whenever t and $t+T$ are in the domain of $f(t)$. For example, the sine and cosine functions are $2\pi-$periodic whereas the tangent and cotangent functions are $\pi-$periodic.

The Laplace transform of a $T-$periodic function is given next.

Theorem 7.4.3

If $f(t)$ is a $T-$periodic, piecewise continuous function for $t \geq 0$ and of exponential order, then

$$\mathcal{L}[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}, \quad s > 0.$$

Proof.

We first have

$$\mathcal{L}[f(t)] = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt.$$

Letting $t = u + T$ we can write

$$\int_T^\infty e^{-st} f(t)dt = \int_0^\infty e^{-s(u+T)} f(u+T)du = e^{-sT} \int_0^\infty e^{-su} f(u)du = e^{-sT}\mathcal{L}[f(t)].$$

Hence,

$$\mathcal{L}[f(t)] = \int_0^T e^{-st} f(t)dt + e^{-sT}\mathcal{L}[f(t)] \Longrightarrow (1 - e^{-sT})\mathcal{L}[f(t)] = \int_0^T e^{-st} f(t)dt.$$

Thus,

$$\mathcal{L}[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}} \blacksquare$$

Example 7.4.10
Determine the Laplace transform of the function

$$f(t) = \begin{cases} 1, & 0 \le t \le \frac{T}{2} \\ 0, & \frac{T}{2} < t < T. \end{cases} \qquad f(t+T) = f(t),\ t \ge 0.$$

Solution.
The graph of $f(t)$ is shown in Figure 7.4.2.

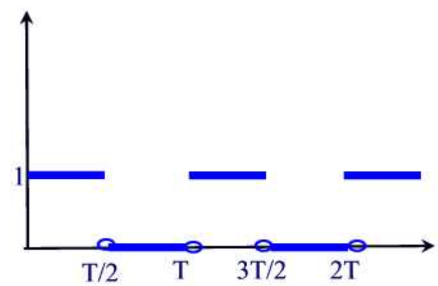

Figure 7.4.2

We have

$$\int_0^T e^{-st} f(t)dt = \int_0^{\frac{T}{2}} e^{-st} dt$$
$$= \frac{1 - e^{-\frac{sT}{2}}}{s}.$$

By Theorem 7.4.3, we have

$$\mathcal{L}[f(t)] = \frac{\frac{1-e^{-\frac{sT}{2}}}{s}}{1 - e^{-sT}} = \frac{1}{s(1 + e^{-\frac{sT}{2}})}, \quad s > 0 \blacksquare$$

Example 7.4.11
Find the Laplace transform of the sawtooth curve shown in Figure 7.4.3

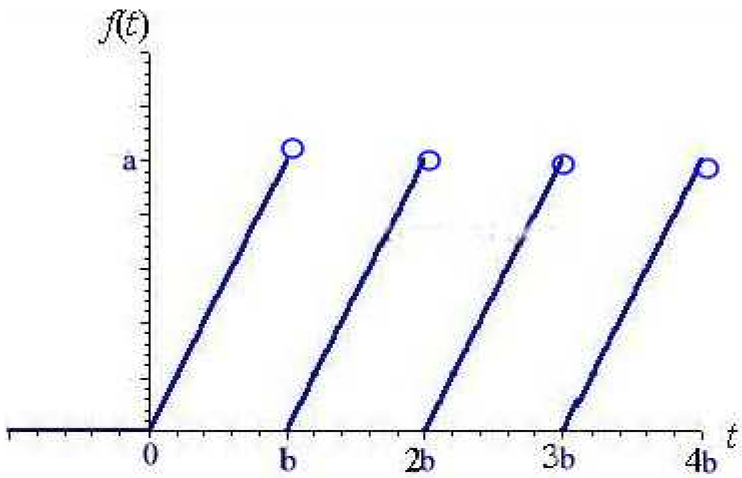

Figure 7.4.3

Solution.
The given function is periodic of period b. For the first period the function is defined by

$$f_b(t) = \frac{a}{b}t[h(t) - h(t-b)].$$

So we have

$$\begin{aligned}\mathcal{L}[f_b(t)] &= \int_0^b e^{-st}\frac{a}{b}tdt \\ &= \frac{a}{b}\left[-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right]_0^b \\ &= \frac{a}{b}\left(\frac{1}{s^2} - \frac{bse^{-bs} + e^{-bs}}{s^2}\right).\end{aligned}$$

Hence,

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[f_b(t)]}{1 - e^{-bs}} = \frac{a}{b}\left[\frac{1 - e^{-bs} - bse^{-bs}}{s^2(1 - e^{-bs})}\right] \blacksquare$$