Arkansas Tech University
MATH 3243: Differential Equations I
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7.3 Shifting Theorems

Properties of the Laplace transform enable us to find Laplace transforms without having to compute them directly from the definition. In this section, we establish properties of Laplace transform that will be useful for solving ODEs.

Shifting Theorems

The next two results are referred to as the first and second shift theorems. As with the linearity property, the shift theorems increase the number of functions for which we can easily find Laplace transforms.

Theorem 7.3.1 *(First Shifting Theorem)*
If $f(t)$ is a piecewise continuous function for $t \geq 0$ and has exponential order then for any real number α we have

$$\mathcal{L}[e^{\alpha t}f(t)] = F(s-\alpha),$$

where $\mathcal{L}[f(t)] = F(s)$.

Proof.
Wew have

$$\mathcal{L}[e^{\alpha t}f(t)] = \int_0^\infty e^{-st}e^{\alpha t}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a) \blacksquare$$

Geometrically, the graph of $F(s-a)$ is obtained by shifting the graph of $F(s)$ a units to the right if $a > 0$ and to the left if $a < 0$.

Example 7.3.1
Evaluate: (a) $\mathcal{L}[e^{2t}t^2]$ (b) $\mathcal{L}[e^{3t}\cos 2t]$.

Solution.
(a) By Theorem 7.3.1, we have $\mathcal{L}[e^{2t}t^2] = F(s-2)$ where $\mathcal{L}[t^2] = \frac{2!}{s^3} = F(s),\ s > 0$. Thus, $\mathcal{L}[e^{2t}t^2] = \frac{2}{(s-2)^3},\ s > 2$.

(b) As in part (a), we have $\mathcal{L}[e^{3t}\cos 2t] = F(s-3)$ where $\mathcal{L}[\cos 2t] = F(s)$. But $\mathcal{L}[\cos 2t] = \frac{s}{s^2+4},\ s > 0$. Thus,

$$\mathcal{L}[e^{3t}\cos 2t] = \frac{s-3}{(s-3)^2+4}, \quad s > 3 \ \blacksquare$$

The next example, illustrates the use of First Shifting in connection with partial fractions decomposition.

Example 7.3.2
Evaluate: (a) $\mathcal{L}^{-1}[\frac{2s+5}{(s-3)^2}]$ (b) $\mathcal{L}^{-1}[\frac{\frac{s}{2}+\frac{5}{3}}{s^2+4s+6}]$.

Solution.
(a) Using the partial fractions decomposition, we can write

$$\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2}.$$

Thus,

$$\mathcal{L}^{-1}[\frac{2s+5}{(s-3)^2}] = 2\mathcal{L}^{-1}[\frac{1}{s-3}] + 11\mathcal{L}^{-1}[\frac{1}{(s-3)^2}] = 2e^{3t} + 11te^{3t}$$

where we used the fact that $F(s) = \frac{1}{s^2} = \mathcal{L}[t]$.
(b) We have

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{\frac{s}{2}+\frac{5}{3}}{s^2+4s+6}\right] =& \mathcal{L}^{-1}\left[\frac{\frac{1}{2}(s+2)+\frac{2}{3}}{(s+2)^2+2}\right] \\
=& \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+2}\right] + \frac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s+2)^2+2}\right] \\
=& \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{2}{3\sqrt{2}}e^{-2t}\sin\sqrt{2}t
\end{aligned}$$

where we used the fact that $\mathcal{L}[\cos\sqrt{2}t] = \frac{s}{s^2+2}$ and $\mathcal{L}[\sin\sqrt{2}t] = \frac{\sqrt{2}}{s^2+2}$ $\blacksquare$

Example 7.3.3
Solve the initial value problem: $y'' - 6y' + 9y = t^2e^{3t},\ y(0) = 2,\ y'(0) = 17$.

Solution.
Taking the Laplace transform of both sides and using the linearity property of $\mathcal{L}$ we can write

$$\mathcal{L}[y''] - 6\mathcal{L}[y'] + 9\mathcal{L}[y] = \mathcal{L}[t^2e^{3t}].$$

Using the Laplace transform of derivatives and the First Shifting Theorem, we obtain

$$s^2Y(s) - sY(0) - Y'(0) - 6[sY(s) - Y(0)] + 9Y(s) = \frac{2}{(s-3)^3}.$$

Using the initial conditions, the above equation can be rearranged as

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

or

$$(s-3)^2Y(s) = 2s + 5 + \frac{2}{(s-3)^3}.$$

Thus,

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5} = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

Hence,

$$\begin{aligned} y(t) =& \mathcal{L}^{-1}\left[\frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}\right] \\ =& 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t} \ \blacksquare \end{aligned}$$

In order to discuss the Second Shifting Theorem, we need to find the Laplace transform of a unit step function.
The Heaviside step function is a piecewise continuous function defined by

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Figure 7.3.1 displays the graph of $h(t)$.

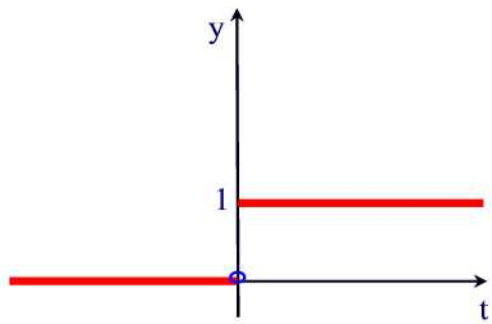

Figure 7.3.1

Taking the Laplace transform of $h(t)$ we find

$$\mathcal{L}[h(t)] = \int_0^\infty h(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s},\ s > 0.$$

A Heaviside function at $\alpha \geq 0$ is the shifted function $h(t-\alpha)$ (α units to the right). For this function, the Laplace transform is

$$\mathcal{L}[h(t-\alpha)] = \int_0^\infty h(t-\alpha)e^{-st}dt = \int_\alpha^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_\alpha^\infty = \frac{e^{-\alpha s}}{s},\ s > 0.$$

Remark 7.3.1
A function of the form

$$f(t) = \begin{cases} f_1(t), & 0 \leq t < a \\ f_2(t), & t \geq a \end{cases}$$

can be written in terms of $h(t)$ as

$$f(t) = f_1(t)[f_2(t) - f_1(t)]h(t-a).$$

Likewise, if

$$f(t) = \begin{cases} 0, & 0 \leq t < a \\ g(t), & a \leq t < b \\ 0, & t \geq b \end{cases}$$

then $f(t) = g(t)[h(t-a) - h(t-b)]$.

Example 7.3.4
Graph the function $f(t) = h(t-1) + h(4-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 2, & 1 \leq t \leq 4 \\ 1, & t > 4. \end{cases}$$

The graph of $f(t)$ is shown in Figure 7.3.2.

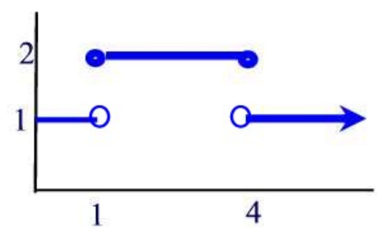

Figure 7.3.2

Thus,

$$\mathcal{L}[f(t)] = \mathcal{L}[h(t-1)] + \mathcal{L}[h(4-t)] = \frac{e^{-s}}{s} + \int_0^4 e^{-st}dt = \frac{1 + e^{-s} - e^{-4s}}{s}, \; s > 0 \blacksquare$$

Theorem 7.3.2 *(Second Shifting Theorem)*
If $F(s) = \mathcal{L}[f(t)]$ and $a > 0$ then

$$\mathcal{L}[f(t-a)h(t-a)] = e^{-as}F(s).$$

Proof.
We have

$$\begin{aligned} \mathcal{L}[f(t-a)h(t-a)] &= \int_0^\infty e^{-st}f(t-a)h(t-a)dt \\ &= \int_0^a e^{-st}f(t-a)h(t-a)dt + \int_a^\infty e^{-st}f(t-a)h(t-a)dt \\ &= \int_a^\infty e^{-st}f(t-a)dt. \end{aligned}$$

Let $v = t - a$, then

$$\int_0^\infty e^{-s(v+a)}f(v)dv = e^{-as}\int_0^\infty e^{-sv}f(v)dve^{-as}\mathcal{L}[f(t)] = e^{-as}F(s) \blacksquare$$

Example 7.3.5
Evaluate: $\mathcal{L}^{-1}[\frac{e^{-2s}}{s^2}]$.

Solution.
Since $\mathcal{L}[t] = \frac{1}{s^2}$, by Theorem 7.3.2, we have

$$\frac{e^{-2s}}{s^2} = \mathcal{L}[(t-2)h(t-2)].$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = (t-2)h(t-2) = \begin{cases} 0, & 0 \le t < 2 \\ t-2, & t \ge 2 \end{cases} \blacksquare$$

Example 7.3.6
Find a formula for $\mathcal{L}[f(t)h(t-a)]$.

Solution.
We have

$$\begin{aligned} \mathcal{L}[f(t)h(t-a)] &= \int_0^\infty e^{-st}f(t)h(t-a)dt \\ &= \int_0^a e^{-st}f(t)h(t-a)dt + \int_a^\infty e^{-st}f(t)h(t-a)dt \\ &= \int_a^\infty e^{-st}f(t)dt = \int_0^\infty e^{-s(v+a)}f(v+a)dv = e^{-as}\mathcal{L}[f(t+a)] \blacksquare \end{aligned}$$

Example 7.3.7
Evaluate: $\mathcal{L}[\cos t h(t-\pi)]$.

Solution.
Using the previous example, we find

$$\mathcal{L}[\cos t h(t-\pi)] = e^{-\pi s}\mathcal{L}[\cos{(t+\pi)}] = e^{-\pi s}\mathcal{L}[-\cos t] = -\frac{se^{-\pi s}}{s^2+1} \blacksquare$$

Example 7.3.8
Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 12, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

Solution.
Note first that $g(t) = 12[h(t-1) - h(t-3)]$ so that

$$\mathcal{L}[g(t)] = 12\mathcal{L}[h(t-1)] - 12\mathcal{L}[h(t-3)] = \frac{12(e^{-s} - e^{-3s})}{s}, \ s > 0.$$

Now taking the Laplace transform of the DE and using linearity we find

$$\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[g(t)].$$

But $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 2$. Letting $\mathcal{L}[y] = Y(s)$ we obtain

$$sY(s) - 2 + 4Y(s) = 12\frac{e^{-s} - e^{-3s}}{s}.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{2}{s+4} + 12\frac{e^{-s} - e^{-3s}}{s(s+4)}.$$

But

$$\mathcal{L}^{-1}\left[\frac{2}{s+4}\right] = 2e^{-4t}$$

and

$$\begin{aligned}
\mathcal{L}^{-1}\left[12\frac{e^{-s} - e^{-3s}}{s(s+4)}\right] =& 3\mathcal{L}^{-1}\left[(e^{-s} - e^{-3s})\left(\frac{1}{s} - \frac{1}{s+4}\right)\right] \\
=& 3\mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+4}\right] + 3\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+4}\right] \\
=& 3h(t-1) - 3h(t-3) - 3e^{-4(t-1)}h(t-1) + 3e^{-4(t-3)}h(t-3).
\end{aligned}$$

Hence,

$$y(t) = 2e^{-4t} + 3[h(t-1) - h(t-3)] - 3[e^{-4(t-1)}h(t-1) - e^{-4(t-3)}h(t-3)], \ t \geq 0 \ \blacksquare$$