Arkansas Tech University
MATH 3243: Differential Equations I
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7.1 The Laplace Transform: Basic Definitions

Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:
• The given "hard" problem is transformed into a "simple" equation.
• This simple equation is solved by purely algebraic manipulations.
• The solution of the simple equation is transformed back to obtain the solution of the given problem.
In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.
The above procedure can be summarized by Figure 7.1.1.

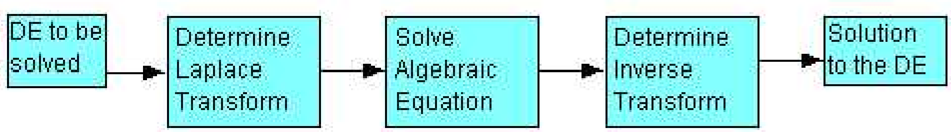

Figure 7.1.1

In this section we introduce the concept of Laplace transform and discuss some of its properties.
The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of f, which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T\to\infty}\int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt.$$

The integral which defines a Laplace transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. When the improper integral is convergent then we say that the function $f(t)$ possesses a Laplace transform. The domain of $F(s)$ depends on the function $f(t)$.
So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral?

Example 7.1.1
Find the Laplace transform, if it exists, of each of the following functions.

$$(a)\ f(t) = e^{at} \quad (b)\ f(t) = 1 \quad (c)\ f(t) = t \quad (d)\ f(t) = \sin{(\omega t)}.$$

Solution.
(a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt.$$

But

$$\int_0^T e^{-(s-a)t}dt = \begin{cases} T & \text{if } s = a \\ \frac{1-e^{-(s-a)T}}{s-a} & \text{if } s \neq a. \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, \quad s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^\infty e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-st}dt = \frac{1}{s},\ \ s > 0.$$

(c) We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2},\ \ s > 0.$$

(d) Using integration by parts, we find

$$\begin{aligned}
\mathcal{L}[\sin{(\omega t)}] =& \int_0^\infty e^{-st}\sin{(\omega t)}dt = \left.\frac{-e^{-st}\sin{(\omega t)}}{s}\right|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos{(\omega t)}dt \\
=& \frac{\omega}{s}\int_0^\infty e^{-st}\cos{(\omega t)}dt = \frac{\omega}{s}\left[\left.\frac{-e^{-st}\cos{(\omega t)}}{s}\right|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin{(\omega t)}dt\right. \\
=& \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\int_0^\infty e^{-st}\sin{(\omega t)}dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\mathcal{L}[\sin{(\omega t)}] \\
\left(1 + \frac{\omega^2}{s^2}\right)\mathcal{L}[\sin{(\omega t)}] =& \frac{\omega}{s^2} \\
\mathcal{L}[\sin{(\omega t)}] =& \frac{\omega}{s^2+\omega^2}, \quad s > 0 \ \blacksquare
\end{aligned}$$

Remark 7.1.1
Similar to part (d), of the previous example, one can easily show that

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, \quad s > 0.$$

Example 7.1.2
Find $\mathcal{L}[t^n],\ n = 1, 2, 3, \cdots$.

Solution.
Let $u' = e^{-st}$ and $v = t^n$. Then $u = -\frac{e^{-st}}{s}$ and $v' = nt^{n-1}$. Hence,

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st} dt = -\left.\frac{t^n e^{-st}}{s}\right|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt, \quad s > 0.$$

That is,

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

Using this, we have

$$\begin{aligned}
\mathcal{L}[t] =& \frac{1}{s^2} \\
\mathcal{L}[t^2] =& \frac{2}{s}\mathcal{L}[t] = \frac{2}{s^3} \\
\mathcal{L}[t^3] =& \frac{3}{s}\mathcal{L}[t^2] = \frac{6}{s^4} \\
\mathcal{L}[t^4] =& \frac{4}{s}\mathcal{L}[t^3] = \frac{24}{s^5} \\
\mathcal{L}[t^5] =& \frac{5}{s}\mathcal{L}[t^4] = \frac{120}{s^5}.
\end{aligned}$$

By induction, one can easily show that for $n = 0, 1, 2, \cdots$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0 \blacksquare$$

The Laplace transform is a linear transform as shown in the next result.

Theorem 7.1.1
Let $f(t)$ and $g(t)$ be two functions that possess a Laplace transform. Then for any scalars α and β we have

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)].$$

Proof.
Suppose that $\mathcal{L}[\alpha f(t)]$ and $\mathcal{L}[\alpha f(t)]$ exist for $s > c$. That is, the improper intergrals $\int_0^t e^{-st} f(t)dt$ and $\int_0^t e^{-st} g(t)dt$ converge for $s > c$. Then the improper integral $\int_0^t e^{-st}[\alpha f(t) + \beta g(t)]dt$ is also convergent and

$$\begin{aligned}\mathcal{L}[\alpha f(t) + \beta g(t)] = & \int_0^t e^{-st}[\alpha f(t) + \beta g(t)]dt \\ = & \alpha \int_0^t e^{-st} f(t)dt + \beta \int_0^t e^{-st} g(t)dt = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)] \blacksquare\end{aligned}$$

Example 7.1.3
Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2\sin(2t)]$. Find the domain of $F(s)$.

Solution.
We have $\mathcal{L}[e^{-7t}] = \frac{1}{s+7},\ s > -7$, $\mathcal{L}[t] = \frac{1}{s^2},\ s > 0$, and $\mathcal{L}[\sin(2t)] = \frac{2}{s^2+4},\ s > 0$. Hence,

$$\mathcal{L}[5e^{-7t} + t + 2\sin(2t)] = 5\mathcal{L}[e^{-7t}] + \mathcal{L}[t] + 2\mathcal{L}[\sin(2t)] = \frac{5}{s+7} + \frac{1}{s^2} + \frac{4}{s^2+4},\ s > 0 \blacksquare$$

Example 7.1.4
Find $\mathcal{L}[\sinh(\omega t)]$ and $\mathcal{L}[\cosh(\omega t)]$.

Solution.
We have

$$\begin{aligned}\mathcal{L}[\sinh(\omega t)] = \mathcal{L}\left[\frac{e^{\omega t} - e^{-\omega t}}{2}\right] = & \frac{1}{2}[\mathcal{L}(e^{\omega t}) - \mathcal{L}(e^{-\omega t})] \\ = & \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{\omega}{s^2 - \omega^2}.\end{aligned}$$

In a similar way, one can show that $\mathcal{L}[\cosh(\omega t)] = \frac{s}{s^2-\omega^2},\quad s > \omega \blacksquare$

The next example provides a function that does not possess a Laplace transform.

Example 7.1.5
Find the Laplace transform, if it exists, of the function $f(t) = e^{t^2}$.

Solution.
Using the definition of Laplace transform we find

$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2-st}dt.$$

If $s \le 0$ then $t^2 - st \ge 0$ so that $e^{t^2-st} \ge 1$ and this implies that $\int_0^\infty e^{t^2-st}dt \ge \int_0^\infty dt$. Since the integral on the right is divergent, by the comparison theorem of improper integrals the integral on the left is also divergent. Now, if $s > 0$ then $\int_0^\infty e^{t(t-s)}dt = \int_0^s e^{t(t-s)}dt + \int_s^\infty e^{t(t-s)}dt$. But $\int_s^\infty e^{t(t-s)}dt \ge \int_s^\infty dt$. By the same reasoning the integral on the left is divergent and so $\int_0^\infty e^{t(t-s)}dt$ is divergent. This shows that the function $f(t) = e^{t^2}$ does not possess a Laplace transform ■

The above example raises the question of what class or classes of functions possess a Laplace transform. Looking closely at Example 7.1.1(a), we notice that for $s > a$ the integral $\int_0^\infty e^{-(s-a)t}dt$ is convergent and a critical component for this convergence is the type of the function $f(t)$. To be more specific, if $f(t)$ is a continuous function such that

$$|f(t)| \le Me^{at}, \quad t \ge C \tag{7.1.1}$$

where $M \ge 0$ and a and C are constants, then this condition yields

$$\int_0^\infty f(t)e^{-st}dt \le \int_0^C f(t)e^{-st}dt + M\int_C^\infty e^{-(s-a)t}dt.$$

Since $f(t)$ is continuous in $0 \le t \le C$, by letting $A = \max\{|f(t)| : 0 \le t \le C\}$ we have

$$\int_0^C f(t)e^{-st}dt \le A\int_0^C e^{-st}dt = A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) < \infty.$$

From the above discussion, we can write

$$|F(s)| \le A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) + \frac{M}{s-a}e^{-(s-a)C}.$$

Hence, $\lim_{s\to\infty} F(s) = 0$.
Also, by Example 7.1.1(a), the integral $\int_C^\infty e^{-(s-a)t}dt$ is convergent for $s > a$.

By the comparison theorem of improper integrals the integral on the left is also convergent. That is, $f(t)$ possesses a Laplace transform.
We call a function that satisfies condition (7.1.1) a function with an **exponential order.** Graphically, this means that the graph of $f(t)$ is contained in the region bounded by the graphs of $y = Me^{at}$ and $y = -Me^{at}$ for $t \geq C$. Note also that this type of functions controls the negative exponential in the transform integral so that to keep the integral from blowing up. Note that f has exponential order if $\lim_{t\to\infty} e^{-at}f(t) = 0$ for some contant a.

Example 7.1.6
(a) Show that $f(t) = t^n$ where n is a positive integer, has an exponential order.
(b) Show that $f(t) = e^{t^2}$ is not of exponential order.

Solution.
(a) Let $a > 0$. Using L'Hôpital's rule n times we can see that $\lim_{t\to\infty} e^{-at}t^n = 0$. Thus, there is a positive constant C such that $e^{-at}t^n \leq 1$ for all $t \geq C$. That is, $t^n \leq e^{at}$ for all $t \geq C$.
(b) Suppose not. That is, let M and C be non-negative constants such that $e^{t^2} \leq Me^{at}$ for some constant a and for all $t \geq C$. Hence, $e^{t^2-at} \leq M$ for all $t \geq C$. Letting $t \to \infty$ we find $M \geq \infty$ which is impossible since $M < \infty$. It follows that $f(t) = e^{t^2}$ is not of exponential order ■

Another question that comes to mind is whether it is possible to relax the condition of continuity on the function $f(t)$. Let's look at the following situation.

Example 7.1.7
Show that the square wave function whose graph is given in Figure 7.1.2 possesses a Laplace transform.

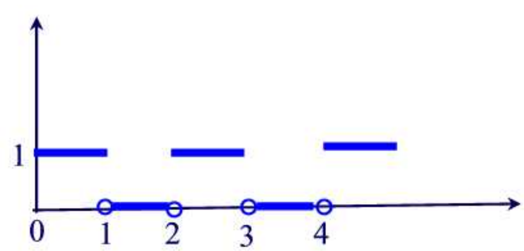

Figure 7.1.2

Note that the function is periodic of period 2.

Solution.
Since $f(t)e^{-st} \leq e^{-st}$, $\int_0^\infty f(t)e^{-st}dt \leq \int_0^\infty e^{-st}dt$. But the integral on the right is convergent for $s > 0$ so that the integral on the left is convergent as well. That is, $\mathcal{L}[f(t)]$ exists for $s > 0$ ■

The function of the above example belongs to a class of functions that we define next.

A function is called **piecewise continuous** on an interval if it consists of a finite number of continuous pieces with possibly either removable or jump discontinuities (but no inifinite discontinuities). Figure 7.1.3 presents a sketch of a piecewise continuous function.

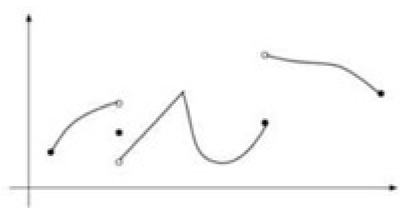

Figure 7.1.3

An example of a function that is not piecewise continuous is the function $f(t) = \frac{1}{t-1}$ on the interval $[0, \infty)$ since at $t = 1$ the continuity is infinite.

Example 7.1.8
Show that the following function is piecewise continuous and of exponentially order.

$$f(t) = \begin{cases} e^{t\sin(t-1)} & \text{if } t \geq 1 \\ \frac{1}{2} & \text{if } 0 \leq t < 1. \end{cases}$$

Solution.
Since $e^{t\sin(t-1)} = (e^{\sin(t-1)})^t \leq (e^1)^t = e^t$ for all $t \geq 0$, $f(t)$ is piecewise continuous and exponentially order ■

The following theorem guarantees the existence of the Laplace transform for all functions that are piecewise continuous and have exponential order.

Theorem 7.1.2 *(Existence)*
Suppose that $f(t)$ is piecewise continuous on $t \geq 0$ and has an exponential order with $|f(t)| \leq Me^{at}$ for $t \geq C$. Then the Laplace transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

exists as long as $s > a$. Note that the two conditions above are sufficient, but not necessary, for $F(s)$ to exist.

Proof.
Using the additive property of integrals, we can write

$$\mathcal{L}[f(t)] = \int_0^C f(t)e^{-st}dt + \int_C^\infty f(t)e^{-st}dt \leq \int_0^C f(t)e^{-st}dt + M\int_C^\infty e^{-(s-a)t}dt.$$

Now, the integral $\int_0^C f(t)e^{-st}dt$ exists being the sum of integrals over intervals where $f(t)e^{-st}$ is continuous. Also,

$$\int_C^\infty e^{-(s-a)t}dt \leq \int_0^\infty e^{-(s-a)t}dt = \mathcal{L}[e^{at}] < \infty, \quad s > a.$$

Hence, by the comparison test of integrals, $\mathcal{L}[f(t)]$ exists ■

Example 7.1.9
Show that the floor function $f(t) = \lfloor t \rfloor$, where for any integer n we have $\lfloor t \rfloor = n$ for all $n \leq t < n+1$ possesses a Laplace transform.

Solution.
The floor function is a piecewise continuous function on $0 \leq t < \infty$. Since $\lfloor t \rfloor \leq t < e^t$ for $0 \leq t < \infty$ we find $M = 1$ and $a = 1$ so that the floor function has an exponential order. Hence, by Theorem 7.1.2, the floor function has a Laplace transform ■

Example 7.1.10
Evaluate $\mathcal{L}[f(t)]$ where

$$f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 1, & t \geq 3 \end{cases}$$

Solution.
We have

$$\mathcal{L}[f(t)] = \int_3^\infty e^{-st}dt = -\left.\frac{e^{-st}}{s}\right|_3^\infty = \frac{e^{-3s}}{s}, s > 0 \ ■$$