Arkansas Tech University
MATH 3243: Differential Equations I
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4.6 Method of Variation of Parameters

In the previous section, we were able to find the general solution to homogeneous linear differential equations with constant coefficients which is the complementary function y_c of the non-homogeneous equation

$$y'' + p(x)y' + q(x)y = g(x). \tag{4.6.1}$$

From Section 4.1, we know that the general solution to the above equation has the structure $y = y_c + y_p$ where y_p is a particular solution to Equation (4.6.1). The purpose of this section is to find y_p. This method has no prior conditions to be satisfied by either $p(x), q(x)$, or $g(x)$.
To use this method, we first find the general solution to the homogeneous equation

$$y_c(x) = c_1y_1(x) + c_2y_2(x).$$

Then we replace the parameters c_1 and c_2 by two functions $u_1(x)$ and $u_2(x)$ to be determined. From this the method got its name. Thus, obtaining

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Observe that if u_1 and u_2 are constant functions then the above y is just the homogeneous solution to the differential equation.
In order to determine the two functions one has to impose two constraints. Finding the derivative of y_p we obtain

$$y_p' = (y_1'u_1 + y_2'u_2) + (y_1u_1' + y_2u_2').$$

Finding the second derivative to obtain

$$y_p'' = y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2' + (y_1u_1' + y_2u_2')'.$$

Since it is up to us to choose u_1 and u_2 we decide to do that in such a way to make our computation simple. One way to achieving that is to impose the condition

$$y_1u_1' + y_2u_2' = 0. \tag{4.6.2}$$

Under such a constraint y_p' and y_p'' are simplified to

$$y_p' = y_1'u_1 + y_2'u_2$$

and

$$y_p'' = y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2'.$$

In particular, y_p'' does not involve u_1'' and u_2''.
Inserting y_p, y_p', and y_p'' into equation (4.6.1) to obtain

$$[y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2'] + p(x)(y_1'u_1 + y_2'u_2) + q(x)(u_1y_1 + u_2y_2) = g(x).$$

Rearranging terms,

$$[y_1'' + p(x)y_1' + q(x)y_1]u_1 + [y_2'' + p(x)y_2' + q(x)y_2]u_2 + [u_1'y_1' + u_2'y_2'] = g(x).$$

Since y_1 and y_2 are solutions to the homogeneous equation, the previous equation yields our second constraint

$$u_1'y_1' + u_2'y_2' = g(x). \tag{4.6.3}$$

Combining equation (4.6.2) and (4.6.3) we find the system of two equations in the unknowns u_1' and u_2'

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t). \end{aligned}$$

Since $\{y_1, y_2\}$ is a fundamental set, the expression $W(x) = y_1y_2' - y_1'y_2$ is nonzero so that one can find unique u_1' and u_2'. Using the method of elimination, these functions are given by

$$u_1'(x) = -\frac{y_2(x)g(x)}{W(x)} \text{ and } u_2'(x) = \frac{y_1(x)g(x)}{W(x)}.$$

Computing antiderivatives to obtain

$$u_1(x) = \int -\frac{y_2(x)g(x)}{W(x)}dx \text{ and } u_2(x) = \int \frac{y_1(x)g(x)}{W(x)}dx.$$

Example 4.6.1
Find the general solution of

$$y'' - y' - 2y = 2e^{-x}$$

using the method of variation of parameters.

Solution.
The characteristic equation $r^2 - r - 2 = 0$ has roots $r_1 = -1$ and $r_2 = 2$. Thus, $y_1(x) = e^{-x}, y_2(x) = e^{2x}$ and $W(x) = 3e^x$. Hence,

$$u_1(x) = -\int \frac{e^{2x} \cdot 2e^{-x}}{3e^x} dx = -\frac{2}{3}x$$

and

$$u_2(x) = \int \frac{e^{-x} \cdot 2e^{-x}}{3e^x} dx = -\frac{2}{9}e^{-3x}.$$

The particular solution is

$$y_p(x) = -\frac{2}{3}xe^{-x} - \frac{2}{9}e^{-x}.$$

The general solution is then given by

$$y(x) = c_1 e^{-x} + c_2 e^{2x} - \frac{2}{3}xe^{-x} - \frac{2}{9}e^{-x} \blacksquare$$

Example 4.6.2
Find the general solution to $(2x-1)y'' - 4xy' + 4y = (2x-1)^2 e^{-x}$ if $y_1(x) = x$ and $y_2(x) = e^{2x}$ form a fundamental set of solutions to the equation.

Solution.
First we rewrite the equation in standard form

$$y'' - \frac{4x}{2x-1}y' + \frac{4}{2x-1}y = (2x-1)e^{-x}.$$

Since $W(x) = (2x-1)e^{2x}$ we find

$$u_1(x) = -\int \frac{e^{2x} \cdot (2x-1)e^{-x}}{(2x-1)e^{2x}} dt = e^{-x}$$

and

$$u_2(x) = \int \frac{x \cdot (2x-1)e^{-x}}{(2x-1)e^{2x}} dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}.$$

Thus,

$$y_p(x) = xe^{-x} - \frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x} = \frac{2}{3}xe^{-x} - \frac{1}{9}e^{-x}.$$

The general solution is

$$y(x) = c_1 x + c_2 e^{2x} + \frac{2}{3}xe^{-x} - \frac{1}{9}e^{-x} \blacksquare$$

Example 4.6.3
Find the general solution to the differential equation $y'' + y' = \ln x,\ x > 0$.

Solution.
The characterisitc equation $r^2 + r = 0$ has roots $r_1 = 0$ and $r_2 = -1$ so that $y_1(x) = 1,\ y_2(x) = e^{-x}$, and $W(x) = -e^{-x}$. Hence,

$$u_1(x) = -\int \frac{e^{-x}\ln x}{-e^{-x}}dx = \int \ln x dx = x\ln x - x$$
$$u_2(x) = \int \frac{\ln x}{-e^{-x}}dx = -\int e^x \ln x dx = -e^x \ln x + \int \frac{e^x}{x}dx$$

Thus,

$$y_p(x) = x\ln x - x - \ln x + e^{-x}\int \frac{e^x}{x}dx$$

and

$$y(x) = c_1 + c_2e^{-x} + x\ln x - x - \ln x + e^{-x}\int \frac{e^x}{x}dx \blacksquare$$

Example 4.6.4
Find the general solution of

$$y'' + y = \frac{1}{2+\sin x}.$$

Solution.
Since the characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$, the general solution of the corresponding homogeneous equation $y'' + y = 0$ is given by

$$y_c(x) = c_1\cos x + c_2\sin x$$

Since $W(x) = 1$ we find

$$u_1(x) = -\int \frac{\sin x}{2+\sin x}dx = -x + \int \frac{2}{2+\sin x}dx$$
$$u_2(x) = \int \frac{\cos x}{2+\sin x}dx = \ln(2+\sin x)$$

Hence, the particular solution is

$$y_p(x) = \sin x\ln(2+\sin x) + \cos x(\int \frac{2}{2+\sin x}dt - x)$$

and the general solution is

$$y(x) = c_1\cos x + c_2\sin x + y_p(x) \blacksquare$$

Example 4.6.5
Find the general solution of

$$y'' - y = \frac{1}{x}.$$

Solution.
The characterisitc equation $r^2 - 1 = 0$ has roots $r_1 = -1$ and $r_2 = 1$ so that $y_1(x) = e^x,\ y_2(x) = e^{-x}$, and $W(x) = -2$. Hence,

$$\begin{aligned} u_1(x) =& \frac{1}{2}\int_{x_0}^{x} \frac{e^t}{t} dt \\ u_2(x) =& -\frac{1}{2}\int_{x_0}^{x} \frac{e^t}{t} dt. \end{aligned}$$

Thus,

$$y_p(x) = \frac{1}{2}e^x \int_{x_0}^{x} \frac{e^t}{t} dt - \frac{1}{2}e^{-x} \int_{x_0}^{x} \frac{e^t}{t} dt$$

and

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{2}e^x \int_{x_0}^{x} \frac{e^t}{t} dt - \frac{1}{2}e^{-x} \int_{x_0}^{x} \frac{e^t}{t} dt \ \blacksquare$$