Arkansas Tech University
MATH 3243: Differential Equations I
Dr. Marcel B Finan

## 4.3 Linear Homogeneous Equations with Constant Coefficients

In Section 4.1, we discussed the structure of the general solution of an $n^{\text{th}}$ order linear homogeneous differential equation. As we saw, the general solution is a linear combination of $n$ solutions that form a fundamental set of solutions. In this section, we discuss a method for finding the fundamental set of solutions for $n^{\text{th}}$ order homogeneous equations with constant coefficients, i.e., equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \cdots, a_n$ are constants with $a_n \neq 0$.
We begin by considering the case $n = 2$, that is the differential equation

$$ay'' + by' + cy = 0 \qquad (4.3.1)$$

where $a, b$ and $c$ are constants with $a \neq 0$. Notice first that for $b = 0$ and $c \neq 0$ the function $y''$ is a constant multiple of $y$. So it makes sense to look for a function with such property. One such function is $y(t) = e^{rx}$. Substituting this function into (4.3.1) leads to

$$ay'' + by' + cy = ar^2 e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx} = 0.$$

Since $e^{rx} > 0$ for all $x$, the previous equation leads to

$$ar^2 + br + c = 0. \qquad (4.3.2)$$

Thus, a function $y(x) = e^{rx}$ is a solution to (4.3.1) when $r$ satisfies equation (4.3.2). We call (4.3.2) the **characteristic equation** for (4.3.1) and the polynomial $C(r) = ar^2 + br + c$ is called the **characteristic polynomial**.

**Example 4.3.1**
Solve: $y'' - 5y' - 6y = 0$.

**Solution.**
The characteristic polynomial for this equation is $C(r) = r^2 - 5r - 6 = (r-2)(r-3)$. Thus, the roots of the characteristic equation are $r = 2$ and $r = 3$. Since

$$W(x) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$$

the functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ form a fundamental set of solutions. Hence, the general solution is given by $y(x) = c_1e^{2x} + c_2e^{3x}$ where $c_1$ and $c_2$ are arbitrary constants ■

We conclude from the previous example that the two distinct real solutions to the characteristic equation lead to the general solution. Does this result apply to any equation (4.3.1) whose characteristic equation has distinct solutions? The answer is in the affirmative. To see this, let $r_1$ and $r_2$ be the two distinct solutions to (4.3.2). Then

$$W(x) = \begin{vmatrix} e^{r_1x} & e^{r_2x} \\ r_1e^{r_1x} & r_2e^{r_2x} \end{vmatrix} = r_2e^{(r_1+r_2)x} - r_1e^{(r_1+r_2)x} = (r_2 - r_1)e^{(r_1+r_2)x} \neq 0$$

since both $r_1 - r_2$ and $e^{(r_1+r_2)x}$ are not equal to 0. Hence, $e^{r_1x}$ and $e^{r_2x}$ form a fundamental set of solutions. As a result, the general solution of (4.3.1) is given by $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$ where $c_1$ and $c_2$ are arbitrary constants.

**Example 4.3.2**
Solve the initial value problem

$$y'' - y' - 6y = 0, \; y(0) = 1, \; y'(0) = 2.$$

**Solution.**
The characteristic polynomial is $C(r) = r^2 - r - 6 = (r-3)(r+2)$ so that the characteristic equation $r^2 - r - 6 = 0$ has the solutions $r_1 = 3$ and $r_2 = -2$. The general solution is then given by

$$y(t) = c_1e^{3x} + c_2e^{-2x}.$$

Taking the derivative to obtain

$$y'(x) = 3c_1e^{3x} - 2c_2e^{-2x}.$$

The conditions $y(0) = 1$ and $y'(0) = 2$ lead to the system

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 - 2c_2 &= 2. \end{aligned}$$

Solving this system by the method of elimination we find $c_1 = \frac{4}{5}$ and $c_2 = \frac{1}{5}$. Hence, the unique solution to the initial value problem is

$$y(x) = \frac{1}{5}(4e^{3x} + e^{-2x}) \blacksquare$$

In the above examples, the characteristic equation has two distinct real roots. If the characterisitic equation has a double root, that is, $r_1 = r_2 = -\frac{b}{2a}$, then $y_1(x) = e^{r_1 x}$ is a solution and using Section 4.2, we can find a second solution

$$y_2(x) = e^{r_1 x} \int \frac{e^{2r_1 x}}{e^{2r_1 x}} dx = xe^{r_1 x}$$

such that $\{y_1, y_2\}$ is a fundamental set of solution. Hence, the general solution is

$$y(x) = c_1 e^{r_1 x} + C_2 x e^{r_1 x}.$$

**Example 4.3.3**
Solve the initial value problem: $y'' + 2y' + y = 0,\ y(0) = 1,\ y'(0) = -1.$

**Solution.**
The characteristic equation $r^2 + 2r + 1 = 0$ has a repeated root: $r_1 = r_2 = -1$. Thus, the general solution is given by

$$y(t) = c_1 e^{-x} + c_2 x e^{-x}.$$

The two conditions $y(0) = 1$ and $y'(0) = -1$ lead to $c_2 = 0$ and $c_1 = 1$. Hence, the unique solution is $y(t) = e^{-x}$ $\blacksquare$.
The characteristic equation can have complex roots. This occurs when $b^2 - 4ac < 0$. In this case the complex roots are

$$r_{1,2} = \alpha \pm i\beta$$

where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$.
Like before, we would like to conclude that the functions

$$e^{(\alpha+i\beta)x} \text{ and } e^{(\alpha-i\beta)x}$$

are solutions to (4.3.1). These are complex solutions, we would like to have real solutions to the original real differential equation. This requires the use of the so-called the **complex exponential function** which we introduce next.
For any complex number $z = \alpha + i\beta$ we define the **Euler's function**

$$e^z = e^\alpha(\cos\beta + i\sin\beta).$$

It follows that the complex solutions to the differential equation are linear combinations of $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$. Now letting $y_1(x) = e^{\alpha x}\cos\beta x$ and $y_2(x) = e^{\alpha x}\sin\beta x$ we find

$$\begin{aligned}
ay_1'' + by_1' + cy_1 =& a(\alpha^2 e^{\alpha x}\cos\beta x - \beta^2 e^{\alpha x}\cos\beta x - 2\alpha\beta e^{\alpha x}\sin\beta x)\\
&+b(\alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x) + ce^{\alpha x}\cos\beta x\\
=& e^{\alpha x}\cos\beta x(a(\alpha^2-\beta^2) + b\alpha + c) - e^{\alpha x}\sin\beta x(2a\alpha\beta + b\beta)\\
=& e^{\alpha x}\cos\beta x\left(a\left(\frac{b^2}{4a^2} - \frac{4ac-b^2}{4a^2}\right) + b\left(\frac{-b}{2a}\right) + c\right)\\
&-e^{\alpha x}\sin\beta x\left(2a\left(-\frac{b}{2a}\frac{\sqrt{4ac-b^2}}{2a}\right) + \frac{b\sqrt{4ac-b^2}}{2a}\right) = 0.
\end{aligned}$$

Thus, $y_1(x) = e^{\alpha x}\cos\beta x$ is a solution to equation (4.3.1). Similarly, we show that $y_2(x) = e^{\alpha x}\sin\beta x$ is a solution to equation (4.3.1). Moreover,

$$W(x) = \begin{vmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x\\ \alpha e^{\alpha X}\cos\beta x - \beta e^{\alpha x}\sin\beta x & \alpha e^{\alpha x}\sin\beta x + \beta e^{\alpha x}\cos\beta x \end{vmatrix} = \beta e^{2\alpha x} \neq 0.$$

Hence, $\{y_1, y_2\}$ is a fundamental set of solutions to equation (4.3.1) so that the general solution is given by

$$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

where $c_1$ and $c_2$ are real numbers.

**Example 4.3.4**
Solve: $y'' + 2y' + 5y = 0$.

**Solution.**
The characteristic equation $r^2 + 2r + 5 = 0$ has complex roots $r_{1,2} = -1 \pm 2i$. The general solution is

$$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) \blacksquare$$

The case $n = 2$ can be generalized to higher order differential equations as illustrated in the next two examples.

**Example 4.3.5**
Solve $y''' + 3y'' - 4y = 0$.

**Solution.**
The associated characteristic equation is $m^3 + 3m^2 - 4 = $ . This can be factored to $(m-1)(m^2 + 4m + 4) = (m-1)(m+2)^2$. Hence, the characteristic roots are $r_1 = 1$ and $r_2 = -2$ of multiplicity 2 so that the general solution is

$$y(x) = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x} \blacksquare$$

**Example 4.3.6**
Solve $y^{(4)} + 2y'' + y = 0$.

**Solution.**
The associated characteristic equation is $m^4 + 2m^2 + 1 = $ . This can be written as $(m^2 + 1)^2 = 0$. Hence, the characteristic roots are $r_1 = -i$ and $r_2 = i$ each of of multiplicity 2. Hence, the general solution is

$$y(x) = c_1 \cos x + c_2 x \cos x + c_3 \sin x + c_4 x \sin x \blacksquare$$