Arkansas Tech University
MATH 3243: Differential Equations I
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4.2 Reduction of Order

In this section, given a non-trivial solution $y_1(x)$ defined on an interval I of the homogeneous ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{4.2.1}$$

we seek a second solution $y_2(x)$ defined on I such that $\{y_1, y_2\}$ is a fundamental set.
Since $a_2(x) \neq 0$, Equation (4.2.1) can be written in the **standard form**

$$y'' + p(x)y' + g(x)y = 0. \tag{4.2.2}$$

The fact that $\{y_1, y_2\}$ is a fundamental set implies that the functions $y_1(x)$ and $y_2(x)$ are linearly independent. Then $\frac{y_2}{y_1}$ is non-constant (for otherwise, we would have $\alpha y_1 - y_2 = 0$ with $\alpha \neq 0$). Hence, $\frac{y_2(x)}{y_1(x)} = u(x)$ for some function $u(x)$. Finding $u(x)$ will result in finding $y_2(x)$. To find $u(x)$, we proceed as follows. Since $y_2(x)$ is supposed to be a solution to Equation (4.2.2), it must satisfy that equation. That is,

$$y_2'' + p(x)y_2' + g(x)y_2 = 0.$$

But $y_2'(x) = u(x)y_1'(x) + u'(x)y_1(x)$ and $y_2''(x) = u(x)y_1''(x) + 2u'(x)y_1'(x) + u''(x)y_1(x)$. Substituting these into the previous equation to obtain

$$u(x)(y_1''(x)+p(x)y_1'(x)+g(x)y_1(x))+u''(x)y_1(x)+u'(x)(2y_1'(x)+p(x)y_1(x)) = 0$$

or

$$u''(x)y_1(x) + u'(x)(2y_1'(x) + p(x)y_1(x)) = 0 \tag{4.2.3}$$

since $y_1(x)$ is a solution to Equation (4.2.2). Now, using the substitution $v(x) = u'(x)$, Equation (4.2.3) becomes

$$v'(x)y_1(x) + v(x)(2y_1'(x) + p(x)y_1(x)) = 0.$$

Separating the variables and integrating to obtain

$$\frac{dv}{v} + 2\frac{dy_1}{y_1} + p(x)dx = 0$$

and

$$\ln|v| + 2\ln|y_1| + \int p(x)dx = C_1.$$

Thus,

$$\ln|vy_1^2| = C_1 - \int p(x)dx$$

or

$$vy_1^2 = C_1 e^{-\int p(x)dx}.$$

From this, we find $u(x)$ by integration

$$u(x) = C_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx + C_2.$$

Choosing $C_1 = 1$ and $C_2 = 0$, we obtain

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2} dx.$$

Example 4.2.1
The function $y_1(x) = x^2$ is a solution to the equation $x^2y'' - 3xy' + 4y = 0$. Find the general solution on the interval $(0, \infty)$.

Solution.
Writing the given equation in standard form, we find

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

so that $p(x) = -\frac{3}{x}$ and $g(x) = \frac{4}{x^2}$. Hence,

$$y_2(x) = x^2 \int \frac{e^{\int \frac{3}{x}dx}}{x^4} dx = x^2 \int \frac{dx}{x} = x^2 \ln x.$$

Hence, the general solution to the differential equation is

$$y(x) = c_1x^2 + c_2x^2 \ln x \blacksquare$$

The reduction of order can be used to find the general solution to the non-homogeneous differential equation

$$a_1(x)y'' + a_2(x)y' + a_1(x)y = g(x)$$

whenever a solution $y_1(x)$ to the associated homogneous equation is known. We illustrate the process in the example below.

Example 4.2.2
The function $y_1(x) = e^{-5x}$ is a solution to the associated homogeneous equation of the differential equation $y'' - 25y = 5$. Use the method of reduction of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given non-homogeneous equation.

Solution.
Let $y(x) = u(x)e^{-5x}$ be a solution to the non-homogeneous equation. Then

$$\begin{aligned} y' =& -5ue^{-5x} + u'e^{-5x} \\ y'' =& 25ue^{-5x} - 10u'e^{-5x} + u''e^{-5x}. \end{aligned}$$

Substituting these into the given differential equation, we find

$$(u'' - 10u')e^{-5x} = 5.$$

Letting $v = u'$, the above equation becomes

$$v' - 10v = 5e^{5x}.$$

Solving by the method of integrating factor with $\mu = e^{-\int 10dx} = e^{-10x}$ we find

$$\begin{aligned} (e^{-10x}v)' =& 5e^{5x}e^{-10x} = 5e^{-5x} \\ e^{-10x}v =& \int 5e^{-5x}dx + c_1 = -e^{-5x} + c_1 \\ v =& -e^{5x} + c_1e^{10x}. \end{aligned}$$

We find u by integrating v,

$$u = \int (-e^{5x} + c_1e^{10x})dx = -\frac{1}{5}e^{5x} + c_1e^{10x} + c_2.$$

Hence,

$$y(x) = c_1e^{5x} + c_2e^{-5x} - \frac{1}{5}$$

which is of the form $y(x) = y_c + y_p$. Hence, $y_2(x) = e^{5x}$ and $y_p = -\frac{1}{5}$ ■