Arkansas Tech University
MATH 3243: Differential Equations I
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4.1 Higher Order Linear ODEs

In this section, we take a look at linear differential equations of order two or more. We start with the following definition: A **linear differential equation of order** n is a differential equation that can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0y = g(x). \qquad (4.1.1)$$

The term "linear" is the result of the fact that the function

$$\mathcal{L}(y) = a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0y$$

satisfies the property

$$\mathcal{L}(\alpha y_1 + \beta y_2) = \alpha\mathcal{L}(y_1) + \beta\mathcal{L}(y_2).$$

If the right-hand side of Equation (4.1.1) is 0 then the equation is said to be **homogeneous**. Otherwise, the equation is said to be **non-homogeneous**. The defining properties of linearity immediately imply the key facts concerning homogeneous linear differential equations.

Theorem 4.1.1
The sum of two solutions to a homogeneous linear differential equation is again a solution, as is the product of a solution by any constant.

Proof.
Let y_1, y_2 be solutions, meaning that $\mathcal{L}(y_1) = 0$ and $\mathcal{L}(y_2) = 0$. Then, thanks to linearity,

$$\mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2) = 0,$$

and hence their sum $y_1 + y_2$ is a solution. Similarly, if α is any constant, and y is any solution, then

$$\mathcal{L}(\alpha y) = \alpha\mathcal{L}(y) = \alpha 0 = 0,$$

and so the scalar multiple αy is also a solution ■

The following result is known as the **superposition principle** for homogeneous linear equations. It states that from given solutions to the equation one can create many more solutions.

Theorem 4.1.2
If $y_1, \cdots, y_n$ are solutions to a common homogeneous linear partial differential equation $\mathcal{L}(y) = 0$, then the **linear combination** $y = c_1y_1 + \cdots + c_ny_n$ is a solution for any choice of constants $c_1, \cdots, c_n$.

Proof.
The key fact is that, thanks to the linearity of $\mathcal{L}$, for any differentiable functions $y_1, \cdots, y_n$ and any constants $c_1, \cdots, c_n$,

$$\begin{aligned}\mathcal{L}(y) =&\mathcal{L}(c_1y_1 + \cdots + c_ny_n) = \mathcal{L}(c_1y_1) + \cdots + \mathcal{L}(c_{n-1}y_{n-1}) + \mathcal{L}(c_ny_n)\\ =&\cdots = c_1\mathcal{L}(y_1) + \cdots + c_{n-1}\mathcal{L}(y_{-1}) + c_n\mathcal{L}(y_n).\end{aligned}$$

Since $\mathcal{L}(y_1) = 0, \cdots, \mathcal{L}(y_n) = 0$, then the right hand side of the above equation vanishes, proving that y is also a solution to the homogeneous equation $\mathcal{L}(y) = 0$ ■

As you have noticed by the above discussion, one or more solutions of a linear homogeneous ODE leads to the creation of lots of solutions according to the Principle of Superposition. In contrast, the Principle of Superposition does not apply to non-homogeneous linear ODEs as shown in the next example.

Example 4.1.1
Consider the differential equation $y' = 1$.
(a) Show that the functions $y_1 = x$ and $y_2 = x + 1$ are solutions to the given differential equation.
(b) Show that the function $y = y_1 + y_2 = 2x + 1$ is not a solution.

Solution.
(a) By simple differentiation we find $y_1' = y_2' = 1$.
(b) Since $y' = 2 \neq 1$, the function y is not a solution ■

As we shall see later, the basic building blocks of the general solution to

$\mathcal{L}(y) = 0$ are linearly independent solutions, a concept that we introduce next.

Linearly Independent and Dependent Functions

We say that the functions $f_1(x), f_2(x), \cdots, f_n(x)$ are **linearly independent** on an interval I if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0,$$

for all x in I is true only when all the $c_i' s$ are equal to zero. If there is at least one $c_i \neq 0$ then we say that the functions are **linearly dependent**.

Example 4.1.2

Show that the functions $f_1(x) = \cos^2 x,\ f_2(x) = \sin^2 x,\ f_3(x) = \sec^2 x$ and $f_4(x) = \tan^2 x$ are linearly dependent on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution.

Since $\cos^2 x + \sin^2 x = 1$ and $\sec^2 x = 1 + \tan^2 x$, we get

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_3 \tan^2 x = 0$$

for all x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where $c_1 = c_2 = 1, c_3 = -1$ and $c_4 = 1$. That is, the given functions are linearly dependent ■

Theorem 4.1.3

The functions $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly dependent if and only if one of the function is a linear combination of the remaining functions.

Proof.

Suppose first that the functions are linearly dependent. Then there are constants $c_1, c_2, \cdots, c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0,$$

for all x in I. Let's say that $c_i \neq 0$. Then we can write

$$\begin{aligned} f_i(x) = & \left(-\frac{c_1}{c_i}\right) f_1(x) + \left(-\frac{c_2}{c_i}\right) f_2(x) + \cdots + \left(-\frac{c_{i-1}}{c_i}\right) f_{i-1}(x) \\ & + \left(-\frac{c_{i+1}}{c_i}\right) f_{i+1}(x) + \cdots + \left(-\frac{c_n}{c_i}\right) f_n(x). \end{aligned}$$

That is, f_i is a linear combination of the remaining functions.
Conversely, suppose that

$$f_i(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_{i-1} f_{i-1}(x) + c_{i+1} f_{i+1}(x) + \cdots c_n f_n(x)$$

for all x in I. Then this can be written as

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0,$$

for all x in I with $c_i = -1 \neq 0$. Hence, $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly dependent ■

It follows from the above theorem, that two functions are linearly dependent if one is a constant multiple of the other or the ratio of the two functions is constant.

Example 4.1.3
Show that the functions $f_1(x) = \sqrt{x} + 5,\ f_2(x) = \sqrt{x} + 5x,\ f_3(x) = x - 1$ and $f_4(x) = x^2$ are linearly dependent on $(0, \infty)$.

Solution.
Since

$$f_2(x) = 1 \cdot f_1(x) + 5 \cdot f_3(x) + 0 \cdot f_4(x)$$

for all x in $(0, \infty)$, the previous theorem asserts that the functions are linearly dependent ■

Alternative Way for Testing Solutions for Independence
Besides the definition introduced above for independence, an alternative way is to use the concept of **Wronskian** which we define next.
Suppose that each of the functions $f_1(x), f_2(x), \cdots, f_n(x)$ has at least $n-1$ derivatives. The **Wronskian** of these functions is the determinant

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Using this concept, we have a criterion for testing the solutions to the homogeneous equation $\mathcal{L}(y) = 0$ for independence.

Theorem 4.1.4
The solutions $y_1, y_2, \cdots, y_n$ of $\mathcal{L}(y) = 0$ are linearly independent on an interval I if $W(y_1, y_2, \cdots, y_n) \neq 0$ for every x in I.

Proof.
The equation

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

leads to the system

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $W(y_1, y_2, \cdots, y_n) \neq 0$ then the matrix with entries the y_is is invertible and this leads to $c_1 = c_2 = \cdots = c_n = 0$. That is, $y_1, y_2, \cdots, y_n$ are linearly independent ■
Notice that if $W(y_1, y_2, \cdots, y_n) = 0$ for every x in I then $y_1, y_2, \cdots, y_n$ are linearly dependent. Thus, for a set of solutions $y_1, y_2, \cdots, y_n$ either $W(y_1, y_2, \cdots, y_n) \neq 0$ for every x in I or $W(y_1, y_2, \cdots, y_n) = 0$ for every x in I. Hence, to show that $y_1, y_2, \cdots, y_n$ are linearly independent it suffices to show that $W(y_1, y_2, \cdots, y_n) = 0$ for some x in I.

Example 4.1.4
Using Wronskian, show that the solutions $y_1(x) = \frac{\cos(2\ln x)}{x^3}$ and $y_2(x) = \frac{\sin(2\ln x)}{x^3}$ of the homogeneous equation

$$x^2y'' + 7xy' + 13y = 0$$

are linearly independent on $(0, \infty)$.

Solution.
We have

$$\begin{aligned} y_1(x) =& \frac{\cos(2\ln x)}{x^3} \\ y_1(1) =& 1 \end{aligned}$$

$$\begin{aligned}
y_1'(x) =& \frac{-2x^2\sin(2\ln x) - 3x^2\cos(2\ln x)}{x^6} \\
y_1'(1) =& -3 \\
y_2(x) =& \frac{\sin(2\ln x)}{x^3} \\
y_2(1) =& 0 \\
y_2'(x) =& \frac{2x^2\cos(2\ln x) - 3x^2\sin(2\ln x)}{x^6} \\
y_2'(1) =& 2.
\end{aligned}$$

Thus,

$$W(y_1(1), y_2(1)) = \begin{vmatrix} 1 & 0 \\ -3 & 2 \end{vmatrix} = 2 \neq 0.$$

Hence, y_1, y_2 are linearly independent $\blacksquare$

Any n linearly independent solution set is called a **fundamental set**. Fundamental sets are the building blocks for finding the general solution to n^{th} order linear homogeneous differential equations.

Theorem 4.1.5
Let $y_1, y_2, \cdots, y_n$ be a fundamental set of $\mathcal{L}(y) = 0$ on an interval I. Then the general solution of the equation on I is given by

$$y = c_1y_1 + c_2y_2 + \cdots + y_n$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

Proof.
We will prove the result for $n = 2$. Let $\{y_1, y_2\}$ be a fundamental set of $a(x)y'' + b(x)y' + c(x)y = 0$. We want to show that every solution y to the differential equation can be written as a linear combination of y_1 and y_2. That is,

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

Since y_1 and y_2 are linearly independent, we can find x_0 in I such that $W(y_1(x_0), y_2(x_0)) \neq 0$.
So the problem reduces to finding the constants c_1 and c_2. These are found

by solving the following linear system of two equations in the unknowns c_1 and c_2:

$$c_1y_1(x_0) + c_2y_2(x_0) = y(x_0)$$
$$c_1y_1'(x_0) + c_2y_2'(x_0) = y'(x_0).$$

By the method of elimination we find

$$c_1 = \frac{y(x_0)y_2'(x_0) - y'(x_0)y_2(x_0)}{W(y_1(x_0), y_2(x_0))}$$

and

$$c_2 = \frac{y'(x_0)y_1(x_0) - y(x_0)y_1'(x_0)}{W(y_1(x_0), y_2(x_0))}.$$

Note that c_1 and c_2 exist since $W(y_1(t_0), y_2(t_0)) \neq 0$ ■

Example 4.1.5
Consider the differential equation

$$y'' + 4y = 0. \tag{4.1.2}$$

(a) Show that $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$ are solutions to (4.1.2).
(b) Show that $\{\cos 2x, \sin 2x\}$ is a fundamental set of solutions.
(c) Write the solution $y(x) = 3\cos\left(2x + \frac{\pi}{4}\right)$ as a linear combination of y_1 and y_2.

Solution.
(a) A simple calculation shows

$$y_1'' + 4y_1 = -4\cos 2x + 4\cos 2x = 0$$
$$y_2'' + 4y_2 = -4\sin 2x + 4\sin 2x = 0.$$

(b) For any x we have

$$W(y_1(x), y_2(x)) = \left|\begin{array}{cc} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{array}\right| = 2\cos^2 2x + 2\sin^2 2x = 2 \neq 0.$$

Thus, $\{y_1, y_2\}$ is a fundamental set of solutions.
(c) Using the formulas for c_1 and c_2 with $x_0 = 0$ we find

$$\begin{aligned} c_1 =& \frac{y(0)y_2'(0) - y'(0)y_2(0)}{W(y_1(0), y_2(0))} \\ =& \frac{6\cos\frac{\pi}{4}\cos 0 + 6\sin\frac{\pi}{4}\sin 0}{2} = \frac{3\sqrt{2}}{2} \end{aligned}$$

and

$$
\begin{aligned}
c_2 =& \frac{y'(0)y_1(0) - y(0)y_1'(0)}{W(y_1(0), y_2(0))} \\
=& \frac{-6\sin\frac{\pi}{4}\cos 0 + 6\cos\frac{\pi}{4}\sin 0}{2} = -\frac{3\sqrt{2}}{2} \blacksquare
\end{aligned}
$$

General Solution to Non-Homogeneous Equation

Next, we consider the question of finding the general solution to the differential equation

$$
a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0 y = g(x) \qquad (4.1.3)
$$

where $g(x) \neq 0$. The following theorem provides the structure of the general solution to equation (4.1.3).

Theorem 4.1.6

Let $\{y_1(x), y_2(x), \cdots, y_n(x)\}$ be a fundamental set of solutions to the homogeneous equation $\mathcal{L}y = 0$ and $y_p(x)$ be a particular solution of the non-homogeneous equation $\mathcal{L}(y_p) = g(x)$. The general solution of the non-homogeneous equation is given by

$$
y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)
$$

for constants $c_1, c_2, \cdots, c_n$.

Proof.

Let $y(x)$ be a solution to the non-homogeneous equation. Let $u(x) = y(x) - y_p(x)$. Then

$$
\mathcal{L}(u) = \mathcal{L}(y) - \mathcal{L}(y_p) = g(x) - g(x) = 0.
$$

Hence, u is a solution to the homogeneous equation so that

$$
u(x) = c_1y_1(x) + c_2y_2(x) + \cdots + y_n(x)
$$

for some constants $c_1, c_2, \cdots, c_n$ and so

$$
y(x) - y_p(x) = c_1y_1(x) + c_2y_2(x) + \cdots + y_n(x)
$$

or

$$
y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) + \cdots + y_n(x) \blacksquare
$$

The general solution to the homogeneous equation $y_c(x) = c_1y_1(x)+c_2y_2(x)+\cdots+y_n(x)$ will be called the **complementary function** for Equation (4.1.3). It follows from the above theorem that in finding the general solution to the non-homogeneous equation, we first solve the associated homogeneous equation and then find a particular solution to the non-homogeneous equation. The general solution to the non-homogenous equation is then the sum of the complementary function and the particular solution.

Example 4.1.6
Consider the differential equation

$$y'' - 2y' - 3y = -9x - 3.$$

(a) Show that $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$ is a fundamental set.
(b) Show that $y_p(x) = 3x - 1$ is a solution to the non-homogeneous equation.
(c) Find the general solution to the non-homogeneous equation.

Solution.
(a) Finding the second derivatives of $y_1(x)$ and $y_2(x)$ we find $y_1'(x) = -e^{-x},\ y_1''(x) = e^{-x},\ y_2'(x) = 3e^{3x},\ y_2'(x) = 9e^{3x}$. Thus,

$$y_1'' - 2y_1' - 3y_1 = e^{-x} + 2e^{-x} - 3e^{-x} = 0$$

and

$$y_2'' - 2y_2' - 3y_2 = 9e^{3x} - 6e^{3x} - 3e^{3x} = 0.$$

Hence, the complementary function is $y_c = c_1e^{-x} + c_2e^{3x}$.
(b) We have

$$y_p'' - 2y_p' - 3y_p = 0 - 6 - 9x + 3 = -9x - 3.$$

(c) The general solution is $y(x) = c_1e^{-x} + c_2e^{3x} + 3x - 1$ ■

As we pointed earlier in the section adding two solutions to a non-homogeneous equation does not necessarily yield a new solution. That is, the Principle of Superposition of homogeneous equations does not hold for non-homogeneous equations. However, we can have a property of superposition of non-homogeneous if one is adding two solutions of two different non-homogeneous equations. More precisely, we have

Theorem 4.1.7
Let $y_1(x)$ be a solution of $a(x)y''+b(x)y'+c(x)y=g_1(x)$ and $y_2(x)$ a solution of $a(x)y''+b(x)y'+c(x)y=g_2(x)$. Then for any constants c_1 and c_2 the function $Y(x)=c_1y_1(x)+c_2y_2(x)$ is a solution of the equation

$$a(x)y''+b(x)y'+c(x)y=c_1g_1(x)+c_2g_2(x).$$

Proof.
We have

$$\begin{aligned}a(x)Y''+b(x)Y'+c(x)Y=&a(x)c_1y_1''+a(x)c_2y_2''+b(x)c_1y_1'+b(x)c_2y_2'+c(x)c_1y_1+c(x)c_2y_2\\=&c_1(a(x)y_1''+b(x)y_1'+c(x)y_1)+c_2(a(x)y_2''+b(x)y_2'+c(x)y_2)\\=&c_1g_1(x)+c_2g_2(x)\ \blacksquare\end{aligned}$$

Example 4.1.7
The functions $u_1(x)$ and $u_2(x)$ are particular solutions to the following differential equations

$$\begin{aligned}a(x)u_1''+b(x)u_1'+c(x)u_1=&2e^{-x}-x-1\\a(x)u_2''+b(x)u_2'+c(x)u_2=&3x.\end{aligned}$$

Use the functions u_1 and u_2 to construct a particular solution of the differential equation

$$a(x)u''+b(x)u'+c(x)u=4e^{-x}-2.$$

Solution.
The right-hand side of the given equation can be written as $4e^{-x}-2=2(2e^{-x}-x-1)+\frac{2}{3}(3x)$ so that by the previous theorem, the function $u(x)=2u_1(x)+\frac{2}{3}u_2(x)$ is the required particular solution $\blacksquare$

Initial Value and Boundary Value Problems
Now, the process of finding y in Equation (4.1.3) requires n integrations which will result in an $n-$parameter family of solutions. In order to find a particular solution, we need to find specific values of the n parameters. In this case, we need n initial conditions such as

$$y(x_0)=y_0,\ y'(x_0)=y_1,\cdots,\ y^{(n-1)}(x_0)=y_{n-1}. \qquad (4.1.4)$$

Equation (4.1.3) subject to conditions (4.1.4) is referred to as an **initial value problem**.

Similar to the existence and uniqueness theorem for first order linear initial value ODEs (Theorem 2.3.2), we have the following existence and uniqueness theorem for higher order version.

Theorem 4.1.8
If $a_0(x), a_1(x), \cdots, a_n(x)$ and $g(x)$ are continuous on an interval I containing x_0 and $a_n(x) \neq 0$ for all x in I then the IVP (4.1.3)-(4.1.4) has a unique solution defined on I.

Example 4.1.8
Show $y(x) = 0$ is the unique solution to the initial value problem

$$y''' + 3y'' - 5y' + 10y = 0, \; y(1) = y'(1) = y''(1) = 0$$

on the interval $(-\infty, \infty)$.

Solution.
Since all the coefficient functions and the right-hand side function are continuous in $(-\infty, \infty)$ and the interval contains 1, Theorem 4.1.8 guarantees the existence of a unique solution. Clearly, $y(x) = 0$ is a solution to the IVP so that it is the only solution ■

The condition $a_n(x) \neq 0$ in the interval I is essential as shown in the next example.

Example 4.1.9
Show $y(x) = cx^2 + x + 3$ is a solution to the initial value problem

$$x^2y'' - 2xy' + 2y = 6, \; y(0) = 3, \; y'(0) = 1$$

on the interval $(-\infty, \infty)$ where c is an arbitrary constant.

Solution.
We have

$$\begin{aligned} x^2y'' - 2xy' + 2y =& x^2(2c) - 2x(2cx + 1) + 2(cx^2 + x + 3) = 6 \\ y(0) =& c(0)^2 + 0 + 3 = 3 \\ y'(x) =& 2cx + 1 \\ y'(0) =& 1. \end{aligned}$$

Thus, the initial value problem has infinite number of solutions. Note that $a_2(0) = 0$ so that the condition $a_n(x) \neq 0$ in Theorem 4.1.8 is violated ■

We conclude this section by introducing another type of linear differential equations subject to certain conditions where the function and its derivatives are specified at different values of x. We limit the discussion to ODE of order 2.
A second order linear ODE

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

subject to the conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1, \ \ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \tag{4.1.5}$$

is called an **boundary value problem**. The conditions (4.1.5) are called **boundary conditions**.
A boundary value problem may have several solutions, one solution or no solution.

Example 4.1.10
The general solution to $x'' + 16x = 0$ is given by $x(t) = c_1 \cos(4t) + c_2 \sin(4t)$.
(a) Find the solution, if it exists, satisfying the conditions $x(0) = x(\frac{\pi}{2}) = 0$.
(b) Find the solution, if it exists, satisfying the conditions $x(0) = x(\pi/8) = 0$.
(c) Find the solution, if it exists, satisfying the conditions $x(0) = 0, \ x(\pi/2) = 1$.

Solution.
(a) From $x(0) = 0$, we find $0 = c_1 \cos 0 = c_1$. Hence, $x(t) = c_2 \sin(4t)$. From the condition $x(\pi/2) = 0$, we find $c_2 \sin(2\pi) = 0$. Since $\sin(2\pi) = 0$, the parameter c_2 can assume any value. Hence, the BVP has infinitely many solutions.
(b) From $x(0) = 0$, we find $0 = c_1 \cos 0 = c_1$. Hence, $x(t) = c_2 \sin(4t)$. From the condition $x(\pi/8) = 0$, we find $c_2 \sin(\pi/2) = 0$ so $c_2 = 0$. Hence, the BVP has the unique solution $y(t) = 0$.
(b) From $x(0) = 0$, we find $0 = c_1 \cos 0 = c_1$. Hence, $x(t) = c_2 \sin(4t)$. From the condition $x(\pi/2) = 1$, we find $c_2 \sin(2\pi) = 1$. Since $\sin(2\pi) = 0$, the value of c_2 does not exist. Hence, the BVP has no solution ■