Arkansas Tech University
MATH 3243: Differential Equations I
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3.2 First Order Non-Linear Models

In this section we investigate some nonlinear models.

Mathematical Ecology: Density-Dependent Population Models

A commonly known principle in the study of population models is the principle of **density-dependence** which states that the growth rate is more likely small for a large population. This makes sense since the larger the population the more scarce the resources become.

Malthus Models

These are Linear models that do not take the density dependence principle into consideration. Instead, the assumption that the same growth characteristics always apply to the population regardless of size. In particular, it is assumed that there are unlimited resources. As a result, linear models are described by the equation $\frac{dP}{dt} = kP$ which always predicts one of two types of behavior: exponential growth for $k > 1$ and exponential decay to 0 or extinction for $0 < k < 1$. Such models are seldom encountered in the real world.

Verhlust or Logistic Models

These are nonlinear models that take into consideration the principle of density-dependence and the effects of population growth. It assumes there is a **carrying capacity** K for the population, i.e., the largest population the environment can sustain. If the population is above K, then the population will decrease, but if below, then it will increase. Consequently, the qualitative behaviors of their solutions are more realistic and reflect the true population dynamics.

A nonlinear model that embodies the density-dependence principle can be described by the first order differential equation

$$\frac{dP}{dt} = Pf(P) \tag{3.2.1}$$

where the growth rate function $f(P)$ decreases as P increases. The simplest type $f(P)$ is a linear one such as

$$f(P) = r\left(1 - \frac{P}{K}\right)$$

where r is a constant called the **growth rate**. Substituting this into Equation (3.1.1), we obtain the following **logistic equation**:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{3.2.2}$$

We call the solution of this differential equation the **logistic function** and its graph the **logistic curve.**

Solving the Logistic Differential Equation

The logistic equation (3.1.2) is a separable differential equation. By separating the variables we find

$$\frac{dP}{P\left(1 - \frac{P}{K}\right)} = rdt.$$

Using, the method of partial fractions one finds

$$\frac{1}{P\left(1 - \frac{P}{K}\right)} = \frac{1}{P} + \frac{1}{K\left(1 - \frac{P}{K}\right)} = \frac{1}{P} + \frac{1}{K - P}.$$

Thus,

$$\frac{dP}{P} + \frac{dP}{K - P} = rdt.$$

Integrating both sides of this equation, we find

$$\int \frac{dP}{P} + \int \frac{dP}{K - P} = \int rdt + C$$

which yields

$$\ln|P| - \ln|K - P| = rt + C$$

or

$$\ln\left|\frac{P}{K - P}\right| = rt + C.$$

Thus,

$$\frac{P}{K - P} = \pm e^{rt+C} = Ce^{rt} \Longrightarrow P(t) = \frac{KCe^{rt}}{1 + Ce^{rt}}.$$

If $P(0) = P_0$ then $P_0 = \frac{KC}{1+C} \Longrightarrow C = \frac{P_0}{K-P_0}$ and so after substituting and simplifying, we find

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}.$$

A logistic curve is shown in Figure 3.2.1.

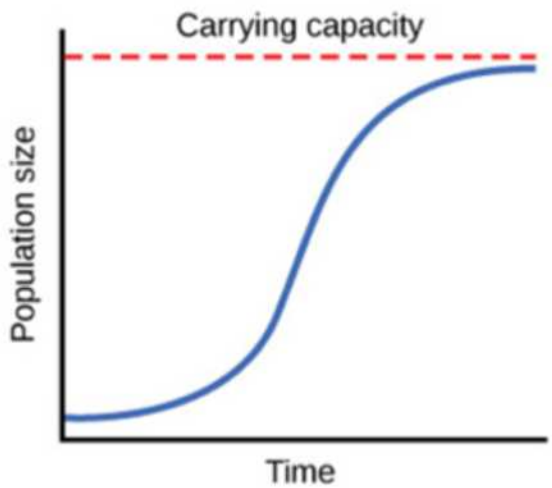

Figure 3.2.1

As is clear from the graph above, a logistic function shows that initial exponential growth is followed by a period in which growth slows and then levels off, approaching (but never attaining) a maximum upper limit which is the carrying capacity. Notice the characteristic S-shape which is typical of logistic functions.

Example 3.2.1
The number $N(t)$ of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem

$$\frac{dN}{dt} = N(1 - 0.0002N), \;\; N(0) = 1.$$

(a) Use the phase portrait concept of Section 2.1 to predict how many supermarkets are expected to adopt the new procedure over a long period of time. By hand, sketch a solution curve of the given initial-value problem.
(b) Solve the initial-value problem.

(c) How many supermarkets are expected to adopt the new technology when $t = 15$? (Round your answer to the nearest integer.)

Solution.
(a) The equilibrium solutions are solutions to the equation $N(1-0.0002N) = 0$. Solving this equation we find $N = 0$ and $N = 5000$. Notice that for $0 < N < 5000$, $\frac{dN}{dt} > 0$. From the phase portrait we see that the equilibrium solution is stable. The solution curve is shown in Figure 3.2.2.

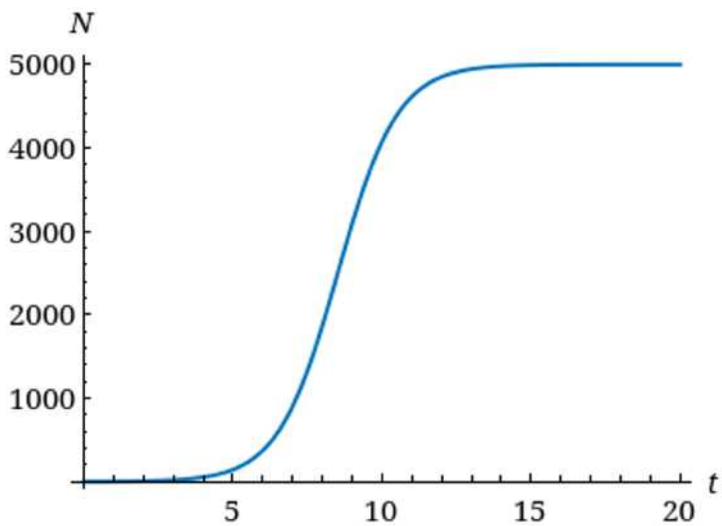

Figure 3.2.2

(b) We have

$$\begin{aligned}
\frac{1}{N(1-0.0002N)} &= \frac{1}{N} + \frac{1}{5000(1-0.0002N)} = \frac{1}{N} + \frac{1}{5000-N} \\
\int \frac{dN}{N(1-0.0002N)} &= \int dt + C \\
\int \frac{dN}{N} + \int \frac{dN}{5000-N} &= t + C \\
\ln|N| - \ln|N-5000| &= t + C \\
\ln\left|\frac{N}{N-5000}\right| &= t + C \\
\frac{N}{N-5000} &= Ce^t.
\end{aligned}$$

Since $N(0) = 1$, we find $C = -\frac{1}{4999}$. Thus,

$$N = (N - 5000)(-1/4999)e^t \Longrightarrow N(t) = \frac{5000e^t}{4999 + e^t}.$$

(c) We have $N(15) = \frac{5000e^{15}}{4999+e^{15}} \approx 4992$ ■