

2.6 Numerical Solutions to ODEs: Euler's Method

Whenever a mathematical problem is encountered in science or engineering, which cannot readily or rapidly be solved by a traditional mathematical method, then a numerical method is usually sought and carried out. In this section, we study Euler's method for approximating the solution to the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a < x < b. \quad (2.6.1)$$

Denote the (unique) solution by $y(x)$. Then $y(x)$ satisfies $y'(x) = f(x, y(x))$. A **numerical method** for solving the initial value problem provides an algorithm for calculating approximate values $y_0, y_1, y_2, \dots, y_n, \dots$ of the solution $y(x)$ at a set of points $x_0 < x_1 < x_2 < \dots < x_n < \dots$. Usually this is a step-by-step process.

Euler's Method

Euler's method is a very simple numerical method for solving the first-order initial value problem (2.6.1). Suppose we want to find approximate values $y_0, y_1, y_2, \dots, y_n, \dots$ of the solution $y(x)$ at equally spaced points $x_0 < x_1 < x_2 < \dots < x_n < \dots$ with step size h . We begin by integrating Equation (2.6.1) between x_n and x_{n+1} obtaining

$$\int_{x_n}^{x_{n+1}} y'(s) ds = \int_{x_n}^{x_{n+1}} f(s, y(s)) ds.$$

By the fundamental theorem of calculus we can write

$$\int_{x_n}^{x_{n+1}} y'(s) ds = y(x_{n+1}) - y(x_n).$$

Thus,

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(s, y(s)) ds. \quad (2.6.2)$$

Computationally, this last equation can not be used since we do not know $y(x)$ and therefore we don't know $f(s, y(s))$ for $x_n \leq s \leq x_{n+1}$. If we estimate the integral by a left Riemann sum we find

$$\int_{x_n}^{x_{n+1}} f(s, y(s)) ds \approx hf(x_n, y(x_n)).$$

Hence,

$$y(x_{n+1}) \approx y(x_n) + hf(x_n, y(x_n)).$$

or

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{2.6.3}$$

where $y_n \approx y(x_n)$. Equation (2.6.3) is known as **Euler's method**. Therefore, we can view Euler's method as a left Riemann sum approximation of the integral equation (2.6.2). Moreover, if we let

$$L(x) = L_0(x) = y_0 + (x - x_0)f(x_0, y(x_0)) = y_0 + (x - x_0)y'(x_0)$$

then (x_1, y_1) is on the tangent line to the solution curve $y(x)$ at (x_0, y_0) . See Figure 2.6.1. Next, y_2 is on the line $L_1(x) = y_1 + (x - x_1)f(x_1, y_1)$. Continuing this process, we find the recursive relation given in Equation (2.6.3). So we Euler's method uses the readily available slope information to start from the point (x_0, y_0) then move from one point to the next along the polygon approximation of the graph of the particular differential equation to ultimately reach the terminal point, (x_n, y_n) .

Example 2.6.1

Suppose that $y(1) = 1$ and $\frac{dy}{dx} = 0.2xy$. Estimate $y(1.5)$ using Euler's method with $h = 0.1$. Note that the exact solution, found by the method of separation of variables, is $y(x) = e^{0.1(x^2-1)}$.

Solution.

The following chart lists the steps needed:

n	x_n	y_n	$f(x_n, y_n)h$
0	1	1	0.0200
1	1.1	1.02	0.0224
2	1.2	1.0424	0.0250
3	1.3	1.0674	0.0278
4	1.4	1.0952	0.03067
5	1.5	1.1259	

Thus, $y(1.5) \approx 1.1259$. Note that the exact value is $y(1.5) = e^{0.1(1.5^2-1)} = 1.1331$ ■

Estimation Error

As with any estimation method, an error is being committed. We define the **absolute error** of the linear approximation to be the difference $E = \text{Exact} - \text{Approximate}$. Figure 2.6.1 shows the approximation of $y(x_1)$ and its absolute error.

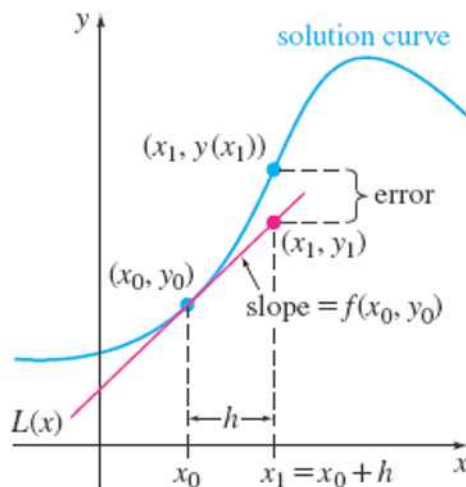


Figure 2.6.1

Absolute error tells us how much our estimate deviates from the correct value or how large the error is. If $E < 0$ then the approximation is an **overestimate** whereas if $E > 0$ then the approximation is an **underestimate**. Note that an absolute error has the same units as the quantities measured.

Two setbacks of this type of error: First, its significance is not very informative. Second, it depends on units. For example, if the exact value is 5 km and the estimate is 4 km then the absolute error is 1 km. If we use meters instead of kilometers then the absolute error would be 1000 m.

A better estimate of error is the **relative error** defined as the ratio of absolute error over actual value. Thus, a relative error tells us how large the absolute error is compared to the correct value. In contrast to absolute value, relative error has no units. The smaller the relative error the better the estimation. When the relative error is written in percentage form, we refer to it as the **percentage relative error**.

Example 2.6.2

Find the absolute error, the relative error and the percentage error of the approximation $y(1.5)$ in Example 2.6.1.

Solution.

The absolute error is

$$E = 1.1331 - 1.1259 = 0.0072.$$

The relative error is

$$\frac{0.0072}{1.1331} = 0.0064$$

and the percentage relative error is 0.64% ■