Arkansas Tech University
MATH 3243: Differential Equations I
Dr. Marcel B Finan

2.4 Exact Differential Equations

We shall now present another technique for solving first order, non-linear, ordinary differential equations. This technique is a generalization of the one we used for separable equations.
We have seen that the solution procedure of separable equations consists of reversing the chain rule. This same procedure works for exact equations but this time the chain rule is for functions of two variables.

The Extended Chain Rule
You recall the chain rule for functions of one variable: If u is differentiable at x and f is differentiable at $u(x)$ then the composite function $y = f(u(x))$ is also differentiable at x with derivative given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 2.4.1
Find the derivative of the function $y = e^{\sqrt{x}}$.

Solution.
Let $u(x) = \sqrt{x}$ and $f(x) = e^x$. Then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and $\frac{dy}{du} = e^u$. Hence,

$$\frac{dy}{dx} = e^u \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \blacksquare$$

The above chain rule can be extended to functions of two variables. Suppose that u and v are differentiable at x and f is a differentiable function of u and v. Then the function $z(x) = f(u(x), v(x))$ is differentiable at x with derivative

$$\frac{dz}{dx} = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}.$$

Example 2.4.2
Let $z = f(u, v) = u^2 + 2u - uv + v^2$ where $u(x) = x^2 + 1$ and $v(x) = x^3 - x^2$. Find $\left.\frac{dz}{dx}\right|_{x=2}$ in two different ways.

Solution.
First notice that $u(2) = 5$ and $v(2) = 4$. By using the extended chain rule we have

$$\begin{aligned}\frac{dz}{dx} =& \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}\\ =&(2u + 2 - v)(2x) + (2v - u)(3x^2 - 2x).\end{aligned}$$

Thus,

$$\left.\frac{dz}{dx}\right|_{x=2} = (10 + 2 - 4)(4) + (8 - 5)(8) = 56.$$

A different way for finding the derivative is to write z as only a function of t obtaining

$$z(x) = x^6 - 3x^5 + 3x^4 - x^3 + 5x^2 + 3.$$

Finding the derivative of $z(x)$

$$z'(x) = 6x^5 - 15x^4 + 12x^3 - 3x^2 + 10x.$$

Finally, $z'(2) = 56$ ■

Exact Differential Equations

The basic idea underlying separable equations is to reverse the chain rule for functions of one variable. The basic idea underlying exact equations is to reverse the extended chain rule. To this end, consider the differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{2.4.1}$$

Let $H(x, y)$ be a function satisfying the two conditions

$$\frac{\partial H}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial H}{\partial y}(x, y) = N(x, y). \tag{2.4.2}$$

Then Equation (2.4.1) can be written as

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}\frac{dy}{dx} = 0. \tag{2.4.3}$$

By the extended chain rule, Equation (2.4.3) is the same as

$$\frac{d}{dx}H(x, y) = 0.$$

Therefore, we obtain an implicitly defined solution given by

$$H(x, y) = C.$$

An equation like (2.4.1) is called **exact** if there is a function $H(x, y)$ satisfying the conditions in (2.4.2).

Testing a Differentiable Equation for Exactness

The next question is the question of telling whether or not Equation (2.4.1) is exact. This is answered by the following theorem.

Theorem 2.4.1

Suppose that the functions $M(x, y)$ and $N(x, y)$ in (2.4.1) are continuous and have continuous first partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in an open rectangle

$$R = \{(x, y) : a < x < b,\ c < y < d\}.$$

Then (2.4.1) is exact in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for all (t, y) in R.

Proof.

Suppose that (2.4.1) is exact in R. Then there is a function $H(x, y)$ such that (2.4.2) is satisfied. Since $M(x, y)$ and $N(x, y)$ are continuous then $\frac{\partial H}{\partial x}(x, y)$ and $\frac{\partial H}{\partial y}(x, y)$ are conitnuous which imply that

$$\frac{\partial^2 H}{\partial y \partial x} = \frac{\partial^2 H}{\partial x \partial y}.$$

Thus,

$$\begin{aligned}\frac{\partial M}{\partial y} =& \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial x}\right) = \frac{\partial^2 H}{\partial y \partial x} \\ =& \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y}\right) = \frac{\partial N}{\partial x}.\end{aligned}$$

Now suppose that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all (x, y) in R. Let $H(x, y)$ be a function such that

$$\frac{\partial H}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial H}{\partial y}(x,y) = N(x,y).$$

Integrating the $M(x,y)$ with respect to x, holding y fixed to obtain

$$H(x,y) = \int M(x,y)dx + f(y). \tag{2.4.4}$$

Differentiating this equation with respect to y, we find

$$N(x,y) = \frac{\partial H}{\partial y}(x,y) = \frac{\partial}{\partial y}\int M(x,y)dx + f'(y)$$

so that

$$f'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx. \tag{2.4.5}$$

Since the left-hand side of this expression is a function of y only, we must show, for consistency, that the right-hand side also depends only on y. Taking the derivative of the right-hand side with respect to x yields

$$\begin{aligned}\frac{\partial}{\partial x}\left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right) =& \frac{\partial N}{\partial x}(x,y) - \frac{\partial^2}{\partial x \partial y}\int M(x,y)dt \\ =& \frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x,y)dx\right) \\ =& \frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = 0\end{aligned}$$

so that the right-hand side of Equation (2.4.5) is a function of y. Now integrating both sides of Equation (2.4.5) with respect to y to find $f(y)$. The expression of $f(y)$ is then being inserted in Equation (2.4.4) ■

Remark 2.4.1
Every separable differential equation is exact. Indeed, since $-g(x)+f(y)y' = 0$ we have $\frac{\partial g}{\partial y} = 0$ and $\frac{\partial f}{\partial x} = 0$. However, not every exact equation is separable. For example, the differential equation $(2x+y)+(2y+x)y' = 0$ is exact since $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$. This equation is clearly not separable ■

Example 2.4.3
Determine whether or not the equation is exact.
(a) $xy^2 + x + x^2yy' = 0$.
(b) $y^2 + 1 + xyy' = 0$.
(c) $\cos y + (y^2 + x\sin y)y' = 0$.
(d) $\cos y + (y^2 - x\sin y)y' = 0$.

Solution.
(a) Since $\frac{\partial}{\partial y}(xy^2 + x) = 2xy$ and $\frac{\partial}{\partial x}(x^2y) = 2xy$, the given equation is exact.
(b) Since $\frac{\partial}{\partial y}(y^2 + 1) = 2y$ and $\frac{\partial}{\partial x}(xy) = y$, the given equation is not exact.
(c) Since $\frac{\partial}{\partial y}(\cos y) = -\sin y$ and $\frac{\partial}{\partial x}(y^2 + x\sin y) = \sin y$, the given equation is not exact.
(d) Since $\frac{\partial}{\partial y}(\cos y) = -\sin y$ and $\frac{\partial}{\partial x}(y^2 - x\sin y) = -\sin y$, the equation is exact ■

Example 2.4.4
Consider the initial value problem

$$x + y + (x + 2y)y' = 0, \quad y(0) = 1.$$

Show that the differential equation is exact and solve the IVP.

Solution.
We have $M(x, y) = x + y$ and $N(x, y) = x + 2y$. Since

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} = 1$$

we have by Theorem 2.4.1 that the differential equation is exact. Thus,

$$H(x, y) = \int (x + y)dx = xy + \frac{x^2}{2} + c_1(y).$$

Hence,

$$x + 2y = \frac{\partial H(x, y)}{\partial y} = x + c_1'(y).$$

It follows that

$$c_1(y) = \int (2y)dy = y^2 + C.$$

Hence,

$$xy + y^2 + \frac{x^2}{2} = C.$$

Since $y(0) = 1$, we find $C = 1$. Thus, y satisfies the implicit equation

$$2xy + 2y^2 + x^2 = 2 \blacksquare$$

Converting a Non-exact DE into Exact

Suppose that Equation (2.4.1) is not exact. Let $\mu(x,y)$ be a function such that

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

is exact.That is,

$$\frac{\partial}{\partial y}(\mu(x,y)M(x,y)) = \frac{\partial}{\partial x}(\mu(x,y)N(x,y)).$$

Using the product rule of differentiation, we find

$$\mu(x,y)\frac{\partial}{\partial y}M(x,y)+\frac{\partial}{\partial y}\mu(x,y)M(x,y) = \mu(x,y)\frac{\partial}{\partial x}N(x,y)+\frac{\partial}{\partial x}\mu(x,y)N(x,y)$$

which can be written in the form

$$(M_y(x,y) - N_x(x,y)\mu(x,y) = \mu_x(x,y)N(x,y) - \mu_y(x,y)M(x,y).$$

This is a partial differential equation, a topic that is not covered in this course. Instead, we are going to consider the follwoing two scenarios:

• If $\mu(x,y) = \mu(x)$ and $\frac{M_y-N_x}{N}$ is a function of x only then $\frac{d\mu}{dx} = \frac{M_y-N_x}{N}\mu$. Solving this equation by separating the variables yields $\mu(x) = e^{\int \frac{M_y-N_x}{N}dx}$. Multiplying the non-exact DE by this function converts it to an exact DE.

• If $\mu(x,y) = \mu(y)$ and $\frac{N_x-M_y}{M}$ is a function of y only then $\frac{d\mu}{dy} = \frac{N_x-M_y}{M}\mu$. Solving this equation by separating the variables yields $\mu(y) = e^{\int \frac{N_x-M_y}{M}dy}$. Multiplying the non-exact DE by this function converts it to an exact DE.

Example 2.4.5

Consider the differential equation

$$xy + (2x^2 + 3y^2 - 20)y' = 0.$$

Show that the differential equation is not exact. With the appropriate integrating factor, convert the DE into an exact differential equation.

Solution.

Since

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y}$$

we choose $\mu(y) = e^{\int \frac{3}{y}dy} = y^3$. Multiplying the given DE by $\mu(y) = y^3$ gives the exact differential equation $xy^4 + (2x^2y^3 + 3y^5 - 20y^3)y' = 0$ ■