Arkansas Tech University
MATH 3243: Differential Equations I
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2.3 First Order Linear ODEs

Any differential equation that can be written in the form

$$y' + p(x)y = g(x) \tag{2.3.1}$$

where $p(x)$ and $g(x)$ are continuous functions with common domain $a < x < b$, is called a **first order linear differential equation**. The term linear is used because $\mathcal{L}(y) = y' + p(x)y$ is linear in y. That is, $\mathcal{L}(\alpha y_1 + \beta y_2) = \alpha\mathcal{L}(y_1) + \beta\mathcal{L}(y_2)$. Indeed, we have

$$\begin{aligned}\mathcal{L}(\alpha y_1 + \beta y_2) =&(\alpha y_1 + \beta y_2)' + p(x)(\alpha y_1 + \beta y_2)\\ =&\alpha y_1' + \alpha p(x)y_1 + \beta y_2' + \beta p(x)y_2\\ =&\alpha(y_1' + p(x)y_1) + \beta(y_2' + p(x)y_2) = \alpha\mathcal{L}(y_1) + \beta\mathcal{L}(y_2).\end{aligned}$$

An ODE that is not linear is called **non-linear.**
In mathematics and physics, linear generally means "simple" and non-linear means "complicated". The theory for solving linear equations is very well developed because linear equations are simple enough to be solvable. Non-linear equations can usually not be solved exactly and are the subject of much on-going research.
Now, we say that Equation (2.3.1) is **homogeneous** if $g(x) \equiv 0$ for all $a < x < b$. If there is a $a < x < b$ such that $g(x) \neq 0$ then Equation (2.3.1) is called **non-homogeneous**. Note that a first order linear homogeneous ODE is also separable ODE.

Example 2.3.1
Classify each of the following first order differential equations as linear or non-linear. If the equation is linear, decide whether it is homogeneous or non-homogeneous.
(a) $\frac{dy}{dx} + \frac{y}{10} = xy$.
(b) $x^2 - 3y^2 + 2x\frac{dy}{dx} = 0$.
(c) $x\frac{dy}{dx} = x^2 - 2y$.
(d) $\frac{dy}{dx} = \frac{x-y}{x+y}$.

Solution.
(a) Notice that the given equation can be written as $\frac{dy}{dx}+(\frac{1}{10}-x)y=0$ which is a homogeneous first order linear DE where $p(x)=\frac{1}{10}-x$ and $g(x)=0$.
(b) This is non-linear because of the term y^2.
(c) This is a non-homogeneous first order linear DE since the right-hand side is not identically zero on any interval. Here, we have $p(x)=\frac{2}{x}$ and $g(x)=x$.
(d) This is non-linear because of the y in the denominator ■

First order linear differential equations possess important linearity or **superposition** properties.

Theorem 2.3.1
(a) If $y_1(x)$ and $y_2(x)$ are any two solutions of the homogeneous equation $y'+p(x)y=0$ then for any constants c_1 and c_2 the linear combination $c_1y_1(x)+c_2y_2(x)$ is also a solution of the homogeneous equation.
(b) If $y_1(x)$ is a solution to the homogeneous equation $y'+p(x)y=0$ and $y_2(x)$ is a solution to the non-homogeneous equation $y'+p(x)y=g(x)$ then $Cy_1(x)+y_2(x)$ is also a solution to the non-homogeneous equation, where C is an arbitrary constant.

Proof.
(a) Since $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation, we have

$$(c_1y_1+c_2y_2)'+p(x)(c_1y_1+c_2y_2)=c_1(y_1'+p(x)y_1)+c_2(y_2'+p(x)y_2)=0+0=0.$$

(b) We have

$$(Cy_1+y_2)'+p(x)(Cy_1+y_2)=C(y_1'+p(x)y_1)+y_2'+p(x)y_2=0+g(x)=g(x)\ ■$$

Remark 2.3.1
Part (a) of the previous theorem is not true in general for non-homogeneous equations. For example, consider the equation $y'=1$. Then $y_1(1)=x$ and $y_2(t)=x+1$ are both solutions to the DE. However, $y_1(x)+y_2(x)=2x+1$ is not a solution since $(y_1+y_2)'=2\neq 1$ ■

Next, we look for the general solution to Equation (2.3.1). The technique we use is a well known technique for solving any first order linear ODE known as the method of **integrating factor**. Let

$$\mu(x)=e^{\int p(x)dx}$$

Multiply Equation (2.3.1) by $\mu(x)$ and notice that the left hand side is just the derivative of $ye^{\int p(x)dx}$. That is,

$$(\mu(x)y)' = \mu(x)g(x).$$

Integrating this last equation to obtain

$$\mu(x)y(x) = \int \mu(x)g(x)dx + C.$$

Thus,

$$y(x) = \frac{1}{e^{\int p(x)dx}} \int e^{\int p(x)dx} g(x)dx + \frac{C}{e^{\int p(x)dx}}. \tag{2.3.2}$$

This is a one-parameter family of solutions.

One can write the above function (2.3.2) in the form $y(x) = Cy_1(x) + y_2(x)$ where $y_1(x) = e^{-\int p(x)dx}$ and $y_2(x) = e^{-\int p(x)dx} \int e^{\int p(x)dx} g(x)dx$. Notice that y_1 is a solution for the homogeneous equation

$$y' + p(x)y = 0.$$

Indeed,

$$y_2' + p(x)y_2 = \left(-\int p(x)dx\right)' e^{-\int p(x)dx} + p(x)e^{-\int p(x)dx} = -p(x)e^{-\int p(x)dx} + p(x)e^{-\int p(x)dx} = 0.$$

Also, y_2 is a particular solution to the non-homogeneous equation. To see this, we let $y_p = e^{-\int p(x)dx} \int e^{\int p(x)dx} g(x)dx$. In this case,

$$\begin{aligned} y_p' + p(x)y_p =& -p(x)e^{-\int p(x)dx} \int e^{\int p(x)dx} g(x)dx + e^{-\int p(x)dx} \cdot e^{\int p(x)dx} g(x) \\ &+ p(x)e^{-\int p(x)dx} \int e^{\int p(x)dx} g(x)dx \\ =& g(x). \end{aligned}$$

Thus, the general solution to Equation (2.3.1) is the sum of a particular solution of the non-homogeneous equation and the general solution of the homogeneous equation which is consistent with Theorem (2.3.1).

Example 2.3.2
Find the general solution to the equation

$$y' + \frac{2}{x}y = \ln x, \ \ x > 0.$$

Solution.
The integrating factor is $\mu(x) = e^{\int \frac{2}{x}dx} = x^2$. Multiplying the given equation by x^2 to obtain

$$(x^2y)' = x^2 \ln x.$$

Integrating with respect to t we find

$$x^2y = \int x^2 \ln x dx + C.$$

The integral on the right-hand side is evaluated using integration by parts with $u = \ln x, dv = x^2dx, du = \frac{dx}{x}, v = \frac{x^3}{3}$ obtaining

$$x^2y = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

Thus,

$$y(x) = \frac{x}{3}\ln x - \frac{x}{9} + \frac{C}{x^2} \blacksquare$$

Next, we look at the conditions that guarantee the existence of a unique solution to the IVP

$$y' + p(x)y = g(x), y(x_0) = y_0. \tag{2.3.3}$$

Theorem 2.3.2
If $p(x)$ and $g(x)$ are continuous functions in the open interval $I = (a, b)$ and x_0 a point inside I then the IVP (2.3.3) has a unique solution $y(x)$ defined on I.

Proof.
Let $F(x, y) = g(x) - p(x)y$. Then $\frac{\partial F}{\partial y}(x, y) = -p(x)$. Hence, $F(x, y)$ and $\frac{\partial F}{\partial y}(x, y)$ are continuous in a rectangle containing (x_0, y_0). By Theorem 1.2.1 of Section 1.2, there is an interval $I_0 \subset I$ containing x_0 such that the IVP (2.3.3) has a unique solution. But when x_0 is in I, finding a solution to (2.3.3) is just a matter of finding an appropriate value of c in (2.3.2). But then the resulting solution is defined for all x is I.That is, the interval of existence of the unique solution is the entire interval I $\blacksquare$

Example 2.3.3
Solve the initial-value problem

$$y' + y = x \quad y(0) = 4.$$

Solution.
By Theorem 2.3.2, the solution exists and is unique on the interval $(-\infty, \infty)$ since 0 belongs to that interval.
We have $p(x) = 1$ so that $\mu(x) = e^x$. Multiplying the given equation by the integrating factor and using the product rule we notice that

$$(e^x y)' = xe^x.$$

Integrating with respect to x and using integration by parts we find

$$e^x y = xe^x - e^x + c.$$

Solving for y we find that the general solution is given by

$$y(x) = x - 1 + ce^{-x}.$$

The condition $y(0) = 4$ implies $c = 5$ and hence the unique solution to the IVP is $y(x) = x-1+5e^{-x},\ -\infty < x < \infty$. Note that for $c \neq 0$, $ce^{-x} \to 0$ as $t \to \infty$. That is, in the long run, all the solutions approach the solution $y = x-1$ corresponding to $c = 0$. In such a case, we call ce^{-x} a **transient term** ■

Remark 2.3.2
Instead of using indefinite integrals in the above discussion one can use definite integrals. For example, replace $\int p(x)dx$ by $\int_{x_0}^{x} p(s)ds$ for some fixed x_0. Using definite integral is proven to be useful when $p(x)$ does not have an elementary function as an antiderivative. For example, when $p(x) = \frac{\sin x}{x}$ or $p(x) = e^{-x^2}$. We illustrate this idea in the next example.

Example 2.3.4
Solve $y' - 2xy = 2,\ y(0) = 1$.

Solution.
Since $p(x) = -2x$, we find $\mu(x) = e^{\int(-2x)dx} = e^{-x^2}$. Thus,

$$\left(e^{-x^2} y\right)' = \left(2\int_0^x e^{-t^2} dt\right)'$$
$$e^{-x^2} y(x) = 2\int_0^x e^{-t^2} dt + C.$$

Since $y(0) = 1$, we find $C = 1$. Hence, $y(x) = e^{x^2} + 2e^{x^2}\int_0^x e^{-t^2}dt$. This last equation can be written in the form

$$y(x) = e^{x^2}\left[1 + \sqrt{\pi}\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt\right)\right] = e^{x^2}[1 + \sqrt{\pi}erf(x)]$$

where $erf(x)$ is known as the **error function** ■

Example 2.3.5
Solve $y' - \frac{1}{x}y = \sin x, \;\; y(1) = 3$. Express your answer in terms of the **sine integral**, $Si(x) = \int_0^x \frac{\sin t}{t} dt$.

Solution.
Since $p(x) = -\frac{1}{x}$, we find $\mu(x) = \frac{1}{x}$. Thus,

$$\begin{aligned}\left(\frac{1}{x}y\right)' =& \frac{\sin x}{x} = \left(\int_0^x \frac{\sin t}{t} dt\right)' \\ \frac{1}{x}y(x) =& Si(x) + C \\ y(x) =& xSi(x) + Cx.\end{aligned}$$

Since $y(1) = 3$, we find $C = 3 - Si(1)$. Hence, $y(x) = xSi(x) + (3 - Si(1))x$ ■

Case when either $p(x)$ or $g(x)$ has a jump discontinuity
Consider the IVP

$$y' + p(x)y = g(x), \quad y(a) = y_0, \quad a \leq x \leq b \tag{2.3.4}$$

where $p(x)$ and $g(x)$ are continuous in $a \leq x \leq b$ except at $t = c$ where either $p(x)$ or $g(x)$ has a jump discontinuity at $a < c < b$. We seek a solution $y(x)$ that is continuous at $x = c$.
To solve this problem, we first solve the initial value problem on the interval $a \leq x < c$ where both $p(x)$ and $g(x)$ are continuous. Let $y_1(x)$ be the unique solution. Since we are seeking a continuous solution to (2.3.4), we expect $y_1(x)$ to have a one-sided limit at c, i.e.,

$$\lim_{x \to c^-} y_1(x) = y_1(c^-).$$

Next, we find the unique solution $y_2(t)$ to the IVP

$$y' + p(x)y = g(x), y(c) = y_1(c^-)$$

where $c \leq x \leq b$. The unique solution to the original IVP is then given by

$$y(x) = \begin{cases} y_1(x), & \text{if } a \leq x < c \\ y_2(x) & \text{if } c \leq x \leq b. \end{cases}$$

Thus, we obtain a peicewise-defined solution. We illustrate this process in the next example.

Example 2.3.6
Find the solution to the IVP

$$y' + \frac{1}{x}y = g(x), \quad y(1) = 1$$

where

$$g(x) = \begin{cases} 3x, & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 2 < x \le 3. \end{cases}$$

The graph of $g(x)$ is given in Figure 2.3.1(a).

Solution.
First, we solve the IVP

$$y' + \frac{1}{x}y = 3x, \quad y(1) = 1, \quad 1 \le x \le 2.$$

The integrating factor is $\mu(x) = x$ and the general solution is $y_1(x) = x^2 + \frac{C}{x}$. Since $y(1) = 1$, we have $C = 0$. Hence, $y_1(x) = x^2$ and $y_1(2) = 4$.
Next, we solve the IVP

$$y' + \frac{1}{x}y = 0, \quad y(2) = 4, \quad 2 < x \le 3.$$

The integrating factor is $\mu(x) = x$ and the general solution is $y_2(x) = \frac{C}{x}$. Since $y_2(2) = 4$ we find $C = 8$. Thus,

$$y(x) = \begin{cases} x^2, & \text{if } 1 \le x \le 2 \\ \frac{8}{x} & \text{if } 2 < x \le 3. \end{cases}$$

The graph of $y(x)$ is given in Figure 2.3.1(b). As you can see from the graph, $y(x)$ is continuous on $[1, 3]$ but not differentiable at $x = 2$ ■

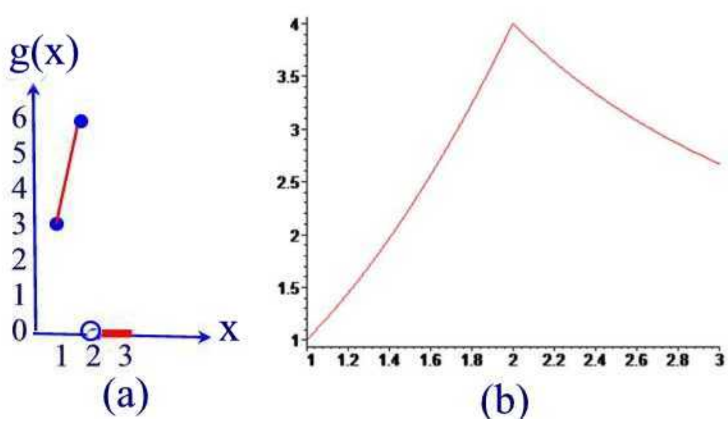

Figure 2.3.1