Arkansas Tech University
MATH 3243: Differential Equations I
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2.2 Separable Differential Equations

In this section, we discuss an analytical method for solving a type of nonlinear first order differential equations, the separable differential equations.
A first order differential equation is **separable** if it can be written with one variable only on the left and the other variable only on the right:

$$f(y)y' = g(x).$$

To solve this equation, we proceed as follows. Let $F(x)$ be an antiderivative of $f(x)$ and $G(x)$ be an antiderivative of $g(x)$. Then by the Chain Rule

$$\frac{d}{dx}F(y) = \frac{dF}{dy}\frac{dy}{dx} = f(y)y'.$$

Thus,

$$f(y)y' - g(x) = \frac{d}{dx}F(y) - \frac{d}{dx}G(x) = \frac{d}{dx}[F(y) - G(x)] = 0.$$

It follows that

$$F(y) - G(x) = C$$

which is equivalent to

$$\int f(y)y'dx = \int g(x)dx + C.$$

As you can see, the result is generally an implicit equation involving a function of y and a function of x. It may or may not be possible to solve this to get y explicitly as a function of x. For an initial value problem, substitute the values of x and y by x_0 and y_0 to get the value of C.

Remark 2.2.1
If F is a differentiable function of y and y is a differentiable function of x and both F and y are given then the chain rule allows us to find $\frac{dF}{dx}$ given by

$$\frac{dF}{dx} = \frac{dF}{dy} \cdot \frac{dy}{dx}.$$

For separable equations, we are given $f(y)y' = \frac{dF}{dx}$ and we are asked to find $F(y)$. This process is referred to as "reversing the chain rule."

Example 2.2.1
Solve the initial value problem $y' = 6xy^2, \quad y(1) = \frac{1}{25}$.

Solution.
Since $f(x, y) = 6xy^2$ and $f_y(x, y) = 12xy$ are continuous in the rectangle

$$R = \{(x, y) : -\infty < x < \infty, \ -\infty < y < \infty\}$$

and R contains the point $\left(1, \frac{1}{25}\right)$, by Theorem 1.2.1, the IVP has a unique solution on some interval containing $x = 1$.
Separating the variables and integrating both sides we obtain

$$\int \frac{y'}{y^2} dx = \int 6x dx + C$$

or

$$-\int \frac{dy}{y^2} = \int 6x dx + C.$$

Thus,

$$-\frac{1}{y(x)} = 3x^2 + C.$$

Since $y(1) = \frac{1}{25}$, we find $C = -28$. The unique solution to the IVP is then given explicitly by

$$y(x) = \frac{1}{28 - 3x^2}.$$

The next question is the question of the interval of existence of this solution. Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, $x = 1$ in this case.
There are three possible intervals where $y(x)$ is continuous:

$$-\infty < x < -\sqrt{\frac{28}{3}}, \quad -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}, \quad x > \sqrt{\frac{28}{3}}.$$

Only one of these will contain the value of x from the initial condition and so we can see that

$$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

must be the interval of existence for this solution. Figure 2.2.1 shows the graph of the solution ■

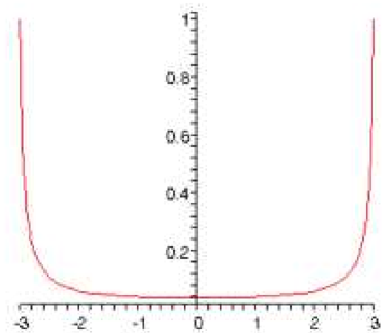

Figure 2.2.1

Example 2.2.2
Solve the IVP $yy' = 4\sin(2x), \quad y(0) = 1$.

Solution.
This is a separable differential equation. Integrating both sides we find

$$\int yy'dx = 4\int \sin(2x)dx + C.$$

Thus,

$$y^2 = -4\cos(2x) + C.$$

Since $y(0) = 1$, we find $C = 5$. Now, solving explicitly for $y(x)$ we find

$$y(x) = \pm\sqrt{-4\cos(2x) + 5}.$$

Since $y(0) = 1$, we find $y(x) = \sqrt{-4\cos(2x) + 5}$. The interval of existence of the solution is the interval $-\infty < x < \infty$ ■

Example 2.2.3
Solve the initial value problem

$$y' = \sqrt{1 - y^2}, \quad y(0) = 0.$$

Solution.
Separating the variables and then integrating we find

$$\int \frac{y'}{\sqrt{1-y^2}} dx = \int dx + C$$

or

$$\int \frac{dy}{\sqrt{1-y^2}} = \int dx + C.$$

Thus,

$$\arcsin y = x + C.$$

Since $y(0) = 0$, we find $C = 0$ and consequently $y(x) = \sin x$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
Now, notice that $y_1(x) = 1$ and $y_1(x) = -1$ are solutions to the differential equations. Moreover, $y(\frac{\pi}{2}) = 1$ and $y(-\frac{\pi}{2}) = -1$ so that the graph of $y = \arcsin x$ is connected to the solution y_1 at the point $x = \frac{\pi}{2}$ and to the solution y_2 at the point $x = -\frac{\pi}{2}$. Thus, the solution to the given IVP is

$$y(x) = \begin{cases} -1, & -\infty < x < -\frac{\pi}{2} \\ \sin x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \infty. \end{cases}$$

The graph of this function is shown in Figure 2.2.2 ■

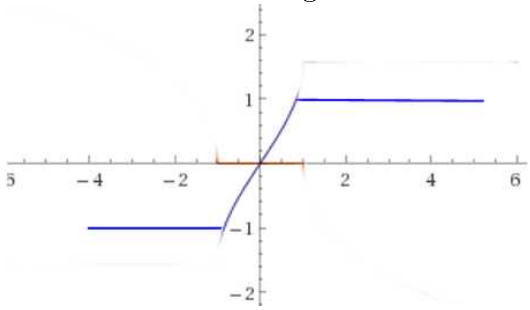

Figure 2.2.2

Example 2.2.4
Solve the (separable) differential equation

$$y' = 2y(2-y).$$

Solution.
Separating the variables and solving (using partial fractions in the process) we find

$$\begin{aligned}\frac{y'}{y(2-y)} =& 2\\ \frac{y'}{2y}+\frac{y'}{2(2-y)} =& 2\\ \frac{1}{2}\int\frac{y'}{y}dx-\frac{1}{2}\int\frac{y'}{y-2}dx =& \int 2dx+C\\ \frac{1}{2}\int\frac{dy}{y}-\frac{1}{2}\int\frac{dy}{y-2} =& \int 2dx+C\\ \ln\left|\frac{y}{y-2}\right| =& 4x+C\\ \left|\frac{y}{y-2}\right| =& Ce^{4x}\\ y(x) =& \frac{2Ce^{4x}}{Ce^{4x}-1}\end{aligned}$$

You should be careful here that the equilibrium solution $y = 2$ can not be obtained from the above formula. That is, $y = 2$ is a singular solution. This solution is lost by dividing by $y - 2$ ■

Example 2.2.5
Solve the initial value problem

$$y' = xy^{\frac{1}{2}},\ y(0) = 0.$$

Solution.
We should first notice that the hypotheses of Theorem 1.2.1 are not satisfied and therefore a unique solution can not be asserted. Using the method of separation of variables, we find

$$\begin{aligned}y^{-\frac{1}{2}}y' =& x\\ \int y^{-\frac{1}{2}}dy =& \int xdx\\ 2y^{\frac{1}{2}} =& \frac{x^2}{2}+C\\ y =& \left(\frac{x^2}{4}+C\right)^{\frac{1}{2}}.\end{aligned}$$

It follows that $y(x) = \frac{x^4}{16}$ is a solution to the IVP. However, the trivial solution $y(x) = 0$ is also a solution which can not be found from the above formula for any values of C. That is, $y(x) = 0$ is a singular solution. This solution is lost by dividing by $y^{\frac{1}{2}}$ ■

It may happen that the implicit solution involves an integral term as the next example illustrates.

Example 2.2.6
Solve the initial value problem

$$y' = e^{-x^2}, \ y(3) = 5.$$

Solution.
Integrating the differential equation from 3 to x, we find $y(x) - y(3) = \int_3^x e^{-t^2} dt$ or $y(x) = 5 + \int_3^x e^{-t^2} dt$ ■