Arkansas Tech University
MATH 3243: Differential Equations I
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2.1 Graphical Solutions to $y' = f(x, y)$

Analytical methods are methods used to find explicit or implicit solutions to differential equations. These methods provide a quantitative analysis of the solutions to differential equations. In many cases explicit or implicit solutions to differential equations are often unobtainable so qualitative analysis can be used to explore features of the solutions to differential equations without the need to solve the differential equation. In this section, we explore methods of finding properties of solutions from the differential equation itself; the principal tool is the geometry of direction field.
A **direction field** (also known as **slope field**) consists of an array of line segments in the xy-plane (called **lineal elements**) having the property that the line plotted at a point (x, y) has slope $f(x, y)$. Direction fields are basically used to visualize the family of solution curves of a given differential equation. In this section we use direction fields for solving initial value problems of the form

$$\frac{dy}{dx} = f(x, y),\ \ y(x_0) = y_0.$$

A single solution in the direction field must follow the flow pattern of the filed; it is tangent to a lineal element when it intersects a point in the grid.
Producing slope fields by hand is a daunting task. For that reason, slope fields are usually created by means of electronic tools. One such tool is the Geogebra (geogebra.org/classic) with learning instructions found at *https : //www.youtube.com/watch?v = G_wVFhJGa1g*

Example 2.1.1
Find the direction field of the differential equation

$$\frac{dy}{dx} = 2x.$$

What is the form of the general solution? Graph the particular solution going through $(0, -1)$.

Solution.
Figure 2.1.1 shows the slope field and the graph of the particular solution to the given DE passing through the point $(0, -1)$. The solution curves look like

parabolas. Thus, the general solution is given by the equation $y = x^2 + C$ ■

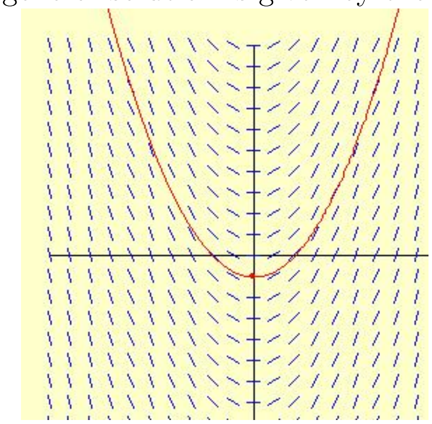

Figure 2.1.1

Example 2.1.2
Using direction field, guess the form of the solution curves of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Solution.
The direction field is given in Figure 2.1.2. The solution curves look like circles centered at the origin. Thus, the general solution is given implicitly by the equation $x^2 + y^2 = C$ where C is a positive constant ■

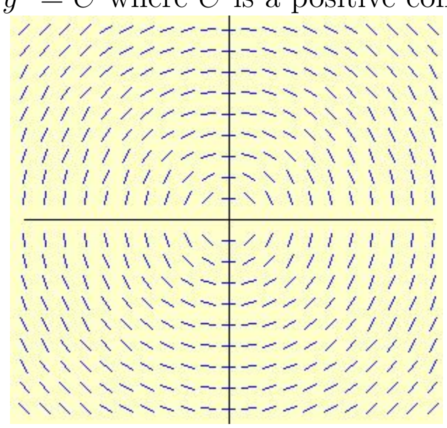

Figure 2.1.2

Remark 2.1.1
We point out here that even though one can draw solution curves, some do not have simple formulas. For instance, the equation $\frac{dy}{dx} = -\frac{1+ye^{xy}}{1+xe^{xy}}$ does not have explicit solutions.

Autonomous First Order Differential Equations

In the special case where $f(x, y) = f(y)$, i.e. the independent variable x does not appear on the right side, the first order DE $\frac{dy}{dx} = f(y)$ is called **autonomous**.

Equilibrium Solutions and Stability for Autonomous Equations

A physical system is often said to be in equilibrium if it doesn't change in time. We adopt this idea and say that a solution to an autonomous first oorder differential equation is an **equilibrium solution**, a **stationary point**, or a **critical point** if it is a constant function. Thus, in a direction field of an autonomous equation, equilibrium solutions are solution curves represented by horizontal lines. It follows that the equations of such solutions have the form $y(x) \equiv c$ where c is a constant. The following result tells us where to look for equilibrium solutions.

Theorem 2.1.1
The function $y(x) \equiv c$, where c is a constant, is an equilibrium solution to $y' = f(y)$ if and only if c is a root of $f(y) = 0$.

Proof.
If the function $y(x) \equiv c$, where c is a constant, is an equilibrium solution to $y' = f(y)$ then it must satisfy the differential equation. This means that $f(c) = 0$. Conversely, if $f(c) = 0$ then the function $y(x) \equiv c$ is a critical point ■

Example 2.1.3
Find the equilibrium solutions to the DE

$$\frac{dy}{dx} = 2y(1 - y).$$

Solution.
The roots of $f(y) = 2y(1 - y) = 0$ are $y = 0$ and $y = 1$. According to the

previous theorem, the equilibrium solutions are $y(x) \equiv 0$ and $y(x) \equiv 1$. The direction field of the DE is shown in Figure 2.1.3 ■

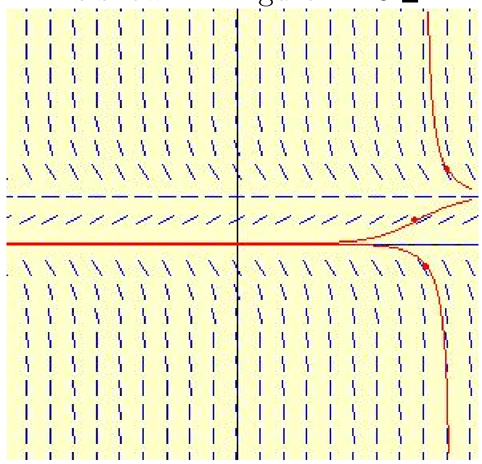

Figure 2.1.3

Remark 2.1.2
Equilibrium solutions can be defined for non-autonomous differential equations. For example, the function $y(x) \equiv 1$ is an equilibrium solution to the DE $y' = (1-y)x^2$.

The direction field of a given differential equation indicates that as x increases without bound, every solution either moves towards or moves away from an equilibrium solution. If all nearby solutions move towards a certain equilibrium solution, then that equilibrium solution is called **asymptotically stable**, **stable**, or **attracting**. The solution $y = 1$ in Figure 2.1.3 is attracting. An equilibrium solution is called **unstable** or **repelling** when all nearby solutions move away from it. The solution $y = 0$ in Figure 2.1.3 is repelling.
If solutions on one side of an equilibrium solution move towards the equilibrium solution and on the other side of the equilibrium solution move away from it then we call the equilibrium solution **semi-stable.**

Example 2.1.4
Sketch the field direction of the differential equation $y' = 4y(1-y)^2$. Show that $y = 1$ is semi-stable.

Solution.
The direction field is shown in Figure 2.1.4. Note that the equilibrium solution $y(x) \equiv 1$ is semi-stable. Nearby solutions that start below it are attracted upward towards it but nearby solutions that start above it are repelled upward and away from it ■

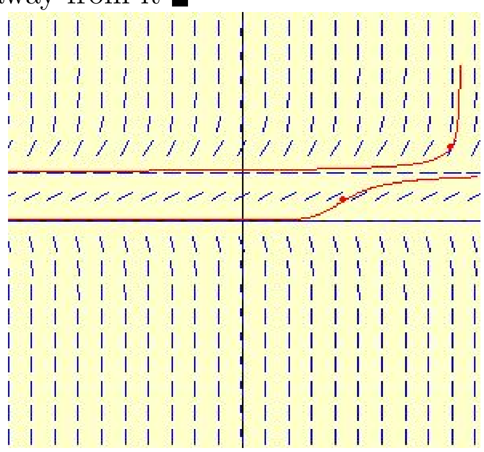

Figure 2.1.4

A very suitable qualitative representation of a differential equation for the study of stability is the so-called phase line. A **phase line** consists of solid dots and arrows. The solid dots represent the equilibrium points and the arrows indicate the directions that solutions move as y increases. Figure 2.1.5(a) shows an example of a phase line. We see that the equilibrium b is stable, whereas the equilibria a and c are unstable.

Example 2.1.5
Consider the autonomous differential equation $\frac{dy}{dt} = f(y)$ where the graph of $f(y)$ is given in Figure 2.1.5(b). Sketch the phase line.

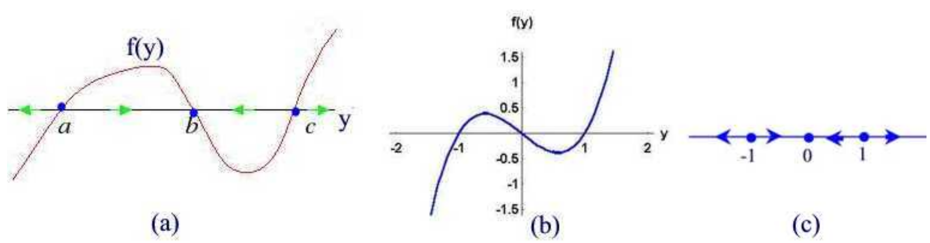

Figure 2.1.5

Solution.
Note that for $y < -1$, $f(y) < 0$ so that the solution $y(x)$ is decreasing. For $-1 < y(x) < 0$, we have $f(y) > 0$ so that $y(x)$ is increasing. For $0 < y(x) < 1$, we have that $f(y) < 0$ so that $y(x)$ is decreasing. Finally, for $y(x) > 1$ we have that $f(y) > 0$ so that $y(x)$ is increasing. Hence, the phase line is given by Figure 2.1.5(c) ■

Remark 2.1.3
Using the existence of unique solutions in Section 1.2, a first order IVP will have a unique solution curve crossing the initial condition point (x_0, y_0). Now, suppose that the differential equation has two equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$. The graphs of these solutions are horizontal lines. Thus, they divide the plane into three subregions, say R_1, R_2, and R_3.
• If (x_0, y_0) is in region R_i then any non-constant solution $y(x)$ crossing (x_0, y_0) remains in R_i and cannot cross the equilibrium solutions. For if $y(x)$ crosses say $y(x) = c_1$ say at a point (x_1, c_1) then the IVP consisting of the differential equation and the intial condition $y(x_1) = c_1$ will have two distinct solutions which contradicts the uniqueness theorem.
• Since f is a continuous function, it can only change sign at a point where $f = 0$, i.e. at an equilibrium point. This can not happen. Hence either $f(y) > 0$ or $f(y) < 0$ for all x in R_i. Thus, $y(x)$ is either always increasing or always decreasing. So $y(x)$ cannot be oscillatory and can not have a maximum or a minimum.
• $y(x)$ will approach an equilibrium solution as either $x \to \infty$ or $x \to -\infty$.