Arkansas Tech University
MATH 3243: Differential Equations I
Dr. Marcel B Finan

1.3 Differential Equations and Mathematical Models

n this section, we look at mathematical models from biology, chemistry, and physics that involve ordinary differential equations.

What is mathematical modeling?
Mathematical modeling is the art of translating problems from an application area into tractable mathematical formulations whose theoretical and numerical analysis provide insight, answers, and guidance useful for the originating application. For example, we use mathematical modeling to provide answers to real world questions like "what is the expected deaths in US from Covid-19 by June 21?". In this section, we look at models from biology, chemistry, and physics.

The Modeling Process
The key steps of the modeling process are as follows:
Step 1: Ask the question.
Step 2: Select the modeling approach.
Step 3: Formulate the model.
Step 4: Solve the model.
Step 5: Answer the question.

Populations Dynamics
If $P(t)$ is the population of a species at time t then by the "conservation of population" law the rate of change of the population is the difference of the rate of population increase, due for example to birth, and the rate of population decrease, due for example to deaths. In mathematical model, we have

$$\frac{dP}{dt} = r_bP - r_dP = (r_b - r_d)P = kP$$

assuming that no migration exists. We call k the **relative growth rate** and is usually given as a percent.
Hence, the above equation says that the rate of change of the population is fixed percent of the population. Thus, the population at time t is given by

$$P(t) = P(0)e^{kt}.$$

Example 1.3.1 *(Doubling Time)*
A certain population grows exponentially. The population grows from 3500 people to 6245 people in 8 years. How long will it take for the original population to double ? This time is called the **doubling time.**

Solution.
We want to find the value of t that will yield a population of 7000 people. So if $7000 = 3500e^{kt}$ then $e^{kt} = 2$. To find k we use the equality $6245 = 3500e^{8k}$. Taking the natural logarithm of both sides we find $8k = \ln\left(\frac{6245}{3500}\right)$ or $k = \frac{1}{8}\ln\left(\frac{6245}{3500}\right) \approx 0.0724$. Thus, $t = \frac{\ln 2}{0.0724} \approx 9.58$ years ■

Radioactive Decay
All materials are made of atoms. Radioactive atoms are unstable; that is, they have too much energy. When radioactive atoms release their extra energy, they are said to decay. All radioactive atoms decay. The rate of change of the mass of the radioactive substance is proportional to the mass present. If $m(t)$ denotes the mass of radioactive substance at time t then by the above statement we have

$$\frac{dm}{dt} = -km, \quad k > 0.$$

Thus, $m(t)$ is given by the formula

$$m(t) = m(0)e^{-kt}, \quad k > 0.$$

Example 1.3.2 *(Half-Life)*
A team of archaeologists thinks they may have discovered Fred Flinstone's fossilized bowling ball. But they want to determine whether the fossil is authentic before they report their discovery to "ABC's Nightline." (Otherwise they run the risk of showing up on "Hard Copy" instead.) Fortunately, one of the scientists is a graduate of ATU's Math 3243, so he calls upon his experience as follows:
The radioactive substance (Carbon 14) has a half-life of 5730 years. By measuring the amount present in a fossil, scientists can estimate how old the fossil is.
Analysis of the "Flinstone bowling ball" determines that 15% of the radioactive substance has already decayed. How old is the fossil ?

Solution.
"Radioactive decay" means that we have a function of the form $m(t) =$

m_0e^{-kt}. Using the given information we can find k. Indeed, $0.5m_0 = m_0e^{-5730k}$. Solving for k we find $k \approx 0.000120968094$. Next, we want to find the desired t. Since $m(t) = .85m_0$ we obtain $m_0e^{-0.000120968094t} = 0.85m_0$. Thus, $t = -\frac{1}{0.000120968094}\ln(0.85) \approx 1343.5$ years ■

Newton's Law of Cooling

Imagine that you are really hungry and in one minute the pizza that you are cooking in the oven will be finished and ready to eat. But it is going to be very hot coming out of the oven. How long will it take for the pizza, which is in an oven heated to 450 degrees Fahrenheit, to cool down to a temperature comfortable enough to eat and enjoy without burning your mouth?

Have you ever wondered how forensic examiners can provide detectives with a time of death (or at least an approximation of the time of death) based on the temperature of the body when it was first discovered?

All of these situations have answers because of Newton's Law of Heating or Cooling. The general idea is that *over time an object will heat up or cool down to the temperature of its surroundings.* The cooling model is given by

$$\frac{dH}{dt} = k(H - S),\ k < 0$$

where S is the temperature of the surroundings. Letting $W = H - S$, the above equation becomes

$$\frac{dW}{dt} = kW$$

whose solution is

$$W(t) = W(0)e^{kt}$$

or

$$H(t) = S + (H(0) - S)e^{kt}.$$

Example 1.3.3

The temperature of a cup of coffee is initially $150°F$. After two minutes its temperature cools to $130°F$. If the surrounding temperature of the room remains constant at $70°F$, how much longer must I wait until the coffee cools to $110°F$?

Solution.

We have

$$H = 70 + (150 - 70)e^{kt} = 70 + 80e^{kt}.$$

To find k we use the fact that $H(2) = 130$. In this case, $130 = 70 + 80e^{2k}$ or $e^{2k} = \frac{3}{4}$. Hence, $k = \frac{1}{2}\ln\frac{3}{4} = -0.144$. To finish the problem we must solve for t in the equation

$$110 = 70 + 80e^{-0.144t}.$$

From this equation, we find $e^{-0.144t} = 0.5$ or $t = \frac{1}{-0.144}\ln 0.5 \approx 4.81$ minutes. Thus, I need to wait an additional 2.81 minutes ■

Spread of a Contagious Disease

A contagious disease such as Covid is spreading among a population. Let $x(t)$ be the number of people infected and $y(t)$ the number of people who have not been yet exposed at a given time t. The rate at which a disease is spreading is proportional to the number of people infected and those who are not yet exposed. Thus, the differential equation for this model is

$$\frac{dx}{dt} = kxy.$$

Mixing Models

All mixing problems we consider here will involve a tank into which a certain mixture will be added at a certain input rate and the mixture will leave the system at a certain output rate. We shall always reserve $y = y(t)$ to denote the amount of substance in the tank at any given time t.

The differential equation involved here arises from the following natural relationship:

$$\frac{dy}{dt} = \textit{input rate} - \textit{output rate}.$$

The main assumption that we will be using here is that the concentration of the substance in the liquid is uniform throughout the tank. Clearly this will not be the case, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course.

Consider a tank initially containing a volume V_0 of mixture (substance and liquid) of concentration c_0. Then the initial amount of the substance is given by $y_0 = c_0V_0$.

Suppose a mixture of concentration $c_i(t)$ flows into the tank at the volume rate $r_i(t)$. Then the substance is entering the tank at the rate $c_i(t)r_i$. Suppose that the well-mixed solution is pumped out of the tank at the volume rate

$r_o(t)$. The concentration of this outflow is $\frac{y(t)}{V(t)}$ where $V(t)$ is the current volume of solution in the tank. Then clearly

$$\frac{dy}{dt} = c_i(t)r_i(t) - \frac{y(t)}{V(t)}r_o(t), \quad y(0) = y_0$$

and

$$\frac{dV}{dt} = r_i(t) - r_o(t).$$

Solving the last equation we find

$$V(t) = V_0 + \int_0^t (r_i(s) - r_o(s))ds.$$

Example 1.3.4
Consider a tank with volume 600 liters containing a salt solution with concentration of $\frac{1}{15}$ $kg/liter$. Suppose a solution with 1/5 kg/liter of salt flows into the tank at a rate of 25 liters/min. The solution in the tank is well-mixed. Solution flows out of the tank at a rate of 50 liters/min. If initially there is 40 kg of salt in the tank then write down the initial value problem for this model.

Solution.
Since the inflow rate is different from the outflow rate then the volume at any time t satisfies $\frac{dV}{dt} = 25 - 50 = -25$ $liters/min$ so that $V(t) = -25t + C$. But $V(0) = 600$ so that $C = 600$. Thus, $V(t) = -25t + 600$. If $y(t)$ is the amount of salt in the tank at any time t then

$$y' = \frac{1}{5} \times 25 - \frac{y}{600 - 25t} \times 50, \quad y(0) = 40$$

or

$$y' + \frac{2y}{24 - t} = 5, \quad y(0) = 40.$$

The above first order linear non-homogeneous differential equation can be solved using the method of integrating factor ■

Falling Bodies with no Air Resistance
Suppose that an object initially at height y_0 is thrown up with initial velocity v_0. Let $y(t)$ denote the distance of the object from the ground , $v(t)$ the

object's velocity, and $a(t)$ the object's acceleration at time t. If air resistance is neglected, then by **Newton's second law**, which states that the net force is equal to the product of mass and acceleration, we have $ma(t) = -mg$. The negative sign on the right-hand of the equation is due to the fact that acceleration due to gravity is pointing downward. Using the fact that $a(t) = y''(t)$ and eliminating the mass, we obtain the equation

$$y'' = -g.$$

To find the velocity $v(t)$ we integrate for a first time and obtain

$$v(t) = y'(t) = -gt + C_1.$$

Since the initial velocity is v_0, we obtain $C_1 = v(0) = v_0$ so that

$$v(t) = -gt + v_0.$$

Integrating for the second time we find the position function

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + C_2.$$

Since y_0 is the initial height, we find $C_2 = y_0$ and so

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0.$$

Example 1.3.5
An object is dropped from the top of a cliff that is 144 feet above ground level.
(a) When will the object reach ground level?
(b) What is the velocity with which the object strikes the ground?

Solution.
(a) The motion of the object translates to the differential equation $y'' = -32$ with solution $y(t) = -16t^2 + 144$. The object reaches ground level when $y(t) = 0$ or $16t^2 = 144$. Solving for t we find $t = \sqrt{\frac{144}{16}} = 3$ sec. The object will reach the ground 3 seconds after it is dropped from the cliff.
(b) The object strikes the ground with velocity $v(3) = -32(3) = -96$ ft/sec ■

Falling Object with Air Resistance

Next, we examine a more realistic model of the one-dimensional motion of an object where we include the effect of air resistance. Air resistance exists because air molecules collide into a falling body creating an upward force opposite gravity and thus reducing the fall of the object. We refer to such a force as the **drag force**.

If we assume that the drag force is proportional to velocity with positive constant of proportionality k then Newton's second law leads to the differential equation

$$m\frac{dv}{dt} = -mg + kv. \tag{1.3.1}$$

Here $k > 0$ depends on the properties of the falling object. Assuming positive direction pointing upward, if the object is moving upward then the drag force is pointing downward and in this case $v < 0$ in Equation (1.3.1). If the object is moving downward then the drag force is pointing upward and in this case $v > 0$ in Equation (1.3.1). Equation (1.3.1) is a first order linear non-homogeneous equation that can be solved using the method of integrating factor.