Arkansas Tech University
MATH 3243: Differential Equations I
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1.2 Existence and Uniqueness of Solutions to First Order IVP

Recall that a solution to an n^{th} order differenital equation requires n integrations to be recovered. This leads to an $n-$parameter family of solutions to the differential equation. To obtain a particular solution at some point x_0, conditions on $y(x_0), y'(x_0), \cdots, y^{n-1}(x_0)$ (known as **initial conditions**), must be given and this leads to finding particular values of the parameters in the family of solutions. The differential equation together with the initial conditions are referred to as the **initial value problem**, abbreviated IVP.

Example 1.2.1
Solve the initial value problem: $y' = y,\ y(0) = 3$.

Solution.
The give differential equation can be written in differential form as $dx - \frac{dy}{y} = 0$. Integrating both sides we find $\int dx - \int \frac{dy}{y} = 0$ or $\ln|y| = x + C$. Solving for y, we find the one-parameter family of solutions $y = Ce^x$. Since $y(0) = 3$, we have $C = 3$ and hence the particular solution to the given IVP is $y = 3e^x$ ■

Example 1.2.2
(a) Show that $y(x) = C_1e^{2x} + C_2e^{-2x}$ is a solution of the differential equation $y'' - 4y = 0$, where C_1 and C_2 are arbitrary constants.
(b) Find a solution satisfying $y(0) = 2$ and $y'(0) = 0$.

Solution.
(a) Finding the first and second derivatives of $y(x)$ to obtain $y'(x) = 2C_1e^{2x} - 2C_2e^{-2x}$ and $y''(x) = 4C_1e^{2x} + 4C_2e^{-2x}$. Thus, $y'' - 4y = 4C_1e^{2x} + 4C_2e^{-2x} - 4(C_1e^{2x} + C_2e^{-2x}) = 0$.
(b) The condition $y(0) = 2$ implies that $C_1 + C_2 = 2$. The condition $y'(0) = 0$ implies that $2C_1 - 2C_2 = 0$ or $C_1 = C_2$. But $C_1 + C_2 = 2$ and this implies that $C_1 = C_2 = 1$. In this case, the particular solution is $y(t) = e^{2x} + e^{-2x}$ ■

When a mathematical model is constructed for physical systems, two reasonable demands are made. First, solutions should exist if the model is to be useful at all. Second, to work effectively in predicting the future behavior of the physical system, the model should produce only one solution for a particular set of initial conditions. Existence and uniqueness theorems help to meet these demands.
In this section we discuss the conditions that guarantee the existence of a unique solution to the initial value problem

$$y' = f(x,y), \; y(x_0) = y_0. \tag{1.2.1}$$

Before we proceed to the major result of this section, we remind the reader of the definition of a partial derivative.

Partial Derivatives
If $f(x,y)$ is a function of two variables x and y then the partial derivative $\frac{\partial f}{\partial x}$ of $f(x,y)$ is the derivative of $f(x,y)$ with respect to x, while pretending y is a constant. The partial derivative $\frac{\partial f}{\partial y}$ is the derivative of $f(x,y)$ with respect to y, while pretending x is constant. The precise definitions are

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

Example 1.2.3
Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y) = x^4y^3 + x^5$.

Solution.
We have

$$\frac{\partial f}{\partial x}(x,y) = 4x^3y^3 + 5x^4 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 3x^4y^2 \;\blacksquare$$

The major existence and uniqueness theorem is stated next.

Theorem 1.2.1
Suppose that $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on the open rectangle

$$R = \{(x,y) : a < x < b, c < y < d\}.$$

Then for any (x_0, y_0) inside R the IVP

$$y' = f(x,y), \quad y(x_0) = y_0$$

has a unique solution defined on an interval of the form $(x_0-h, x_0+h) \subset (a,b)$ for some positive h.

Remark 1.2.1
The conditions in Theorem 1.2.1 are sufficient but not necessary. If the conditions of Theorem 1.2.1 are not satisfied then this does not mean that the problem does not have a unique solution. In fact, if the hypotheses of Theorem 1.2.1 are not met then anything could happen: Problem (1.2.1) may still have a unique solution, or may have several solutions, or it may have no solution at all.

Example 1.2.4
Consider the differential equation

$$y' = \frac{y^{\frac{1}{3}}}{x(y-2)}.$$

Does the existence theorem guarantee the existence of a unique solution to the following IVPs: (a) $y(3) = 4$ (b) $y(0) = 7$ (c) $y(0) = 2$ (d) $y(1) = 2$?

Solution.
The function $f(x, y) = \frac{y^{\frac{1}{3}}}{x(y-2)}$ is continuous for $x \neq 0$ and $y \neq 2$. The function

$$\frac{\partial f}{\partial y}(x, y) = \frac{-2 - 2y}{3x(y-2)^2 y^{\frac{2}{3}}}$$

is continuous for $x \neq 0$ and $y \neq 0, 2$. Thus, f and $\frac{\partial f}{\partial y}$ are continuous for $x \neq 0$ and $y \neq 0, 2$. If we choose R to be the rectangle $0 < x < \infty,\ 2 < y < \infty$ thenTheorem 1.2.1 guarantees the existence of a unique solution for the initial value problem in (a). Since the hypoteses of Theorem 1.2.1 can not be satisfied in any rectangle containing the points $(0, 7)$, there is no guarantee that there is a unique solution or any solution for the IVP in (b). Similar arguments for (c) and (d) ■

Example 1.2.5
Find the largest rectangle where a unique solution to the IVP

$$y' = \frac{y^2}{1 - x^2}, \quad y(0) = 0$$

exists.

Solution.
We have

$$f(x,y) = \frac{y^2}{1-x^2}$$

and

$$\frac{\partial f}{\partial y}(x,y) = \frac{2y}{1-x^2}.$$

These are both continuous functions as long as we avoid the lines $x = \pm 1$. Theorem 1.2.1 tells us that we can expect one and only one solution of

$$y' = \frac{y^2}{1-x^2}, \quad y(0) = 0$$

in the strip

$$R = \{(x,y) : -1 < x < 1, -\infty < y < \infty\} \blacksquare$$

Example 1.2.6
Consider the IVP

$$y' = \frac{1}{2}(-x + \sqrt{x^2+4y}), \; y(2) = -1.$$

(a) Show that $y(x) = 1 - x$ and $y(x) = -\frac{x^2}{4}$ are two solutions to the above IVP.
(b) Does this contradict Theorem 1.2.1?

Solution.
(a) You can verify that the two functions are solutions by substitution.
(b) Since $f(x,y) = \frac{1}{2}(-x + \sqrt{x^2+4y})$ and $f_y(x,y) = \frac{1}{\sqrt{x^2+4y}}$, these two functions are not continuous on any rectangles containing $(2,-1)$. Thus, we can not apply Theorem 1.2.1 for this problem $\blacksquare$

Example 1.2.7
Consider the IVP

$$xy' = y, \; y(0) = 1.$$

Show that this initial value problem has no solution.

Solution.
We have $\frac{dy}{y} = \frac{dx}{x}$ so upon integration, we find $y(x) = Cx$. There is no value of C that yields a function with the condition $y(0) = 1$. Notice that the conditions of Thereom 1.2.1 are not satisfied since $f(x,y) = \frac{y}{x}$ is not continuous in any rectangle containing $(0,1)$ $\blacksquare$

Remark 1.2.2
The interval of existence to an IVP may be larger than the interval $(x_0 - h, x_0 + h)$ of Theorem 1.2.1. Thus, it is best to think that Theorem 1.2.1 guarantees a unique solution defined locally, i.e. near the point (x_0, y_0).

Example 1.2.8
Consider the initial value problem: $x^2y' - y^2 = 0, \quad y(1) = 1.$
(a) Determine the largest open rectangle in the $xy-$plane, containing the point $(x_0, y_0) = (1, 1)$, in which the hypotheses of Theorem 1.2.1 are satisfied.
(b) A solution of the initial value problem is $y(x) = x$. This solution exists on $-\infty < x < \infty$. Does this fact contradicts Theorem 1.2.1? Explain your answer.

Solution.
(a) We have $f(x, y) = \frac{y^2}{x^2}, \; f_y(x, y) = \frac{2y}{x^2}$. So

$$R = \{(x, y) : 0 < x < \infty, \; -\infty < y < \infty\}.$$

(b) No. Theorem 1.2.1 is a local existence theorem and not a global one ■

Example 1.2.9
Is it possible to find a function $f(x, y)$ that is continuous and has continuous partial derivatives in the entire $xy-$plane such that the functions $y_1(x) = \cos x$ and $y_2(x) = 1 - \sin x$ are both solutions to the initial value problem $y' = f(x, y), \; y\left(\frac{\pi}{2}\right) = 0$?

Solution.
Since f is continuous and has continuous partial derivatives in the entire xy-plane, the equation $y' = f(x, y)$ satisfies the conditions of Theorem 1.2.1. Notice that $y_1(\frac{\pi}{2}) = y_2(\frac{\pi}{2}) = 0$, so the curves $y_1(x) = \cos x$ and $y_2(x) = 1 - \sin x$ have a common point $(\frac{\pi}{2}, 0)$, so if they were both solutions of our equation, by the uniqueness theorem they would have to agree on any rectangle containing $(\frac{\pi}{2}, 0)$. Since they do not, they cannot both be solutions of the equation $y' = f(x, y)$ ■

Example 1.2.10
Find the interval of solution of the solution to the IVP: $y' + 2xy^2 = 0, \; y(0) = -1$.

Solution.
The given ODE can be written in the form $\frac{y'}{y^2} = -2x$. Thus, $\int \frac{y'}{y^2} dx = \int (-2x) dx$ which gives $-\frac{1}{y} = -x^2 + C$ or $y(x) = \frac{1}{x^2+C}$. From $y(0) = -1$, we find $C = -2$ so that $y(x) = \frac{1}{x^2-1}$. As a function, the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. As a solution to the differential equation, the domain can be $(-\infty, -1), (-1, 1)$, or $(1, \infty)$. As a solution to the IVP, the domain is $(-1, 1)$ since this interval conatains the initial condition $x = 0$. Hence, the interval of existence of the solution to the IVP is $(-1, 1)$ ■

Boundary Value Problems
Initial conditions are prescribed at a single value of the independent variable. However, in some cases we require that conditions are prescribed at multiple values of the independent variable. In this case, these conditions are called **boundary value conditions**. Boundary value conditions together with the differential equation form what is called **boundary value problems**.

Example 1.2.11
Determine whether the problem is an initial value problem or a boundary value problem.
(a) $y' + 2xy^2 = 0,\ y(0) = -1$.
(b) $y'' + y = 0,\ y(0) = 0,\ y(\pi/6) = 4$.

Solution.
(a) Initial value problem.
(b) Boundary value problem ■